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Performance Evaluation for Model-Based Networked Control Systems

Luis A. Montestruque[1] and Panos J. Antsaklis[2]

[1] EmNet, LLC, Granger, IN, U.S.A. lmontest@heliosware.com
[2] Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN, U.S.A. pantsakl@nd.edu

The performance of a class of Model-Based Networked Control System (MB-NCS) is considered in this paper. A MB-NCS uses an explicit model of the plant to reduce the network bandwidth requirements. In particular, an Output Feedback MB-NCS is studied. After reviewing the stability results for this system and some lifting techniques basics, two performance measures related to the traditional H2 performance measure for LTI systems are computed. The first H2 like performance measurement is called the Extended H2 norm of the system and is based on the norm of the impulse response of the MB-NCS at time zero. The second performance measure is called the Generalized H2 norm and it basically replaces the traditional trace norm by the Hilbert-Schmidt norm that is more appropriate for infinite dimensional operators. The Generalized H2 norm also represents the average norm of the impulse response of the MB-NCS for impulse inputs applied at different times. Examples show how both norms converge to the traditional H2 norm for continuous H2 systems. Finally, with the help of an alternate way of representing lifted parameters, the relationship between the optimal sampler and hold of a sampled data system and the structure of the Output Feedback MB-NCS is shown.

1 Introduction

A networked control system (NCS) is a control system in which a data network is used as feedback media. NCS is an important area see for example [6] and [5, 7, 8]. Industrial control systems are increasingly using networks as media to interconnect the different components. The use of networked control systems poses, though, some challenges. One of the main problems to be addressed when considering a networked control system is the size of bandwidth required by each subsystem. Since each control subsystem must share the same medium the reduction of the individual bandwidth is a major concern. Two ways of addressing this problem are: minimizing the frequency of transfer of information between the sensor and the controller/actuator; or compressing or reducing the size of the data transferred at each transaction. A shared characteristic among popular digital industrial networks are the small transport time and big overhead per packet, thus using fewer bits per packet has small impact over the overall bit rate. *So reducing the rate at which packets are transmitted brings better benefits than data compression in terms of bit rate used*. The MB-NCS architecture makes explicit use of knowledge about the plant dynamics to enhance the performance of the system. Model-Based Networked Control Systems (MB-NCS) were introduced in [1].

Consider the control of a continuous linear plant where the state estimated by a standard observer is sent to a linear controller/actuator via a network. In this case, the controller and observer uses an explicit model of the plant that approximates the plant dynamics and makes possible the stabilization of the plant even under slow network conditions. The controller makes use of a plant model, which is updated with the observer estimate, to reconstruct the actual plant state in between updates. The model state is then used to generate the control signal. *The main idea is to perform the feedback by updating the model's state using the observer estimated state of the plant. The rest of the time the control action is based on a plant model that is incorporated in the controller/actuator and is running open loop for a period of h seconds.* Also a disturbance signal w and a performance or objective signal z are included in the setup. The control architecture is shown in Figure **1**.

The observer has as inputs the output and input of the plant. In the implementation, in order to acquire the input of the plant, which is at the other side of the communication link, the observer can have a version of the model and controller, and knowledge of the update time h. In this way, the output of the controller, that is the input of the plant, can be simultaneously and continuously generated at both ends of the feedback path with the only requirement being that the observer makes sure that the controller has been updated. This last requirement ensures that both the controller and the observer are synchronized. Handshaking protocols provided by most networks can be used.

The performance characterization of Networked Control Systems under different conditions is of major concern. In this paper a class of networked control systems called Model-Based Networked Control Systems (MB-NCS) is considered. This control architecture uses an explicit model of the plant in order to reduce the network traffic while attempting to prevent excessive performance degradation. MB-NCS can successfully address several important control issues in an intuitive and transparent way. Necessary and sufficient stability results have been reported for continuous and discrete linear time invariant systems with state feedback, output feedback, and with network-induced delays (see [1, 2, 3]). Results for stochastic update-times have also been derived [4]. We have observed that the stability of MB-NCS is, in general, a function of the update times, the difference between the dynamics of the plant and the dynamics of the plant model, and of the control law used,

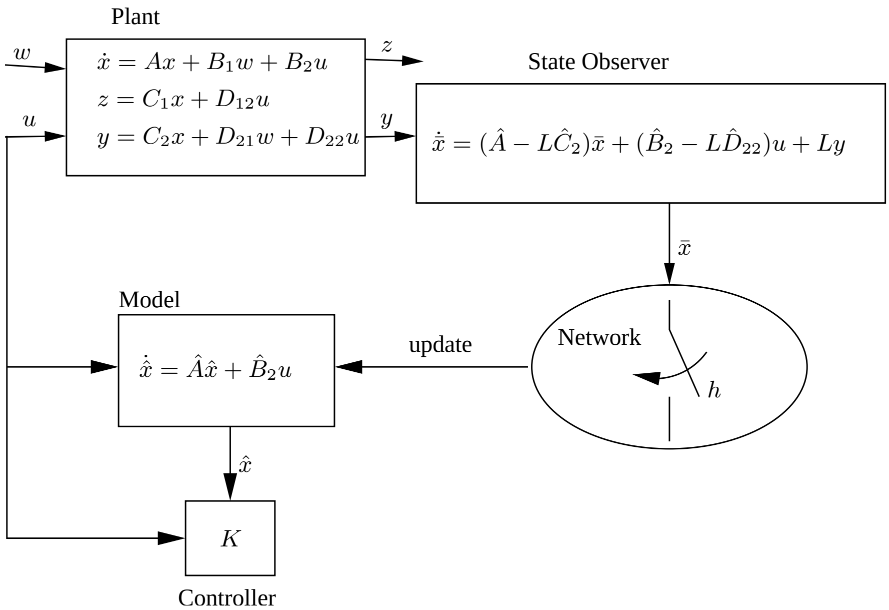

Fig. 1. MB-NCS with disturbance input and objective signal output.

and we have quantified these relations. The performance of the MB-NCS can be studied using several techniques and considering different scenarios. One promising technique is called Continuous Lifting [9, 10, 11]. Lifting is basically a transformation of a periodic system to a discrete LTI system. The main advantage of this approach is that most of the results available for LTI systems are readily applicable to the lifted system. The disadvantage is that the input and output spaces are infinite dimensional and thus the parameters of the lifted system are operators and not matrices. New results in this area allow overcoming these difficulties [12, 13].

The next section briefly introduces the lifting techniques used to derive the results contained in the paper. Then, an H2 like performance measure is introduced as the Extended H2 norm, here the interplay between the discrete and continuous nature of the system is observed in the calculation of the norm. Also, a way of calculating the Extended H2 norm using an auxiliary LTI discrete system is presented. The next section presents the Generalized H2 norm. This norm is important because it can also be related to the norm of a operator based transfer function. Again, a way of computing the norm using an auxiliary LTI discrete system is derived. The paper is finalized with a discussion of the techniques described in [12, 13] and their application to optimal control synthesis problems for data sampled systems. This allows to efficiently compute the optimal gains for the controller and observer. Not surprisingly it is shown that under certain conditions, the optimal gains for the H2 optimal observer and controller of the MB-NCS are equivalent to the optimal gains for the non-networked system.

We will start by defining the system dynamics:

$$\begin{array}{l} \text{Plant Dynamics:} \\ \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{12} u \\ y = C_2 x + D_{21} w + D_{22} u \\ \text{Observer Dynamics:} \\ \dot{\bar{x}} = (\hat{A} - L\hat{C}_2)\bar{x} + (\hat{B}_2 - L\hat{D}_{22})u + Ly \\ \text{Model Dynamics:} \\ \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}_2 u \\ \text{Controller:} \\ u = K\hat{x} \end{array} \tag{1}$$

The model state $\hat{x}$ is updated with the observer state $\bar{x}$ every h seconds. It can be shown that the system dynamics can be described by:

$$\begin{array}{l} G_{zw}: \\ \begin{bmatrix} \dot{x} \\ \dot{\bar{x}} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A & B_2 K & -B_2 K \\ LC_2 & \hat{A} - L\hat{C}_2 + \hat{B}_2 K + L\tilde{D}_{22} K & -\hat{B}_2 K - L\tilde{D}_{22} K \\ LC_2 & L\tilde{D}_{22} K - L\hat{C}_2 & \hat{A} - L\tilde{D}_{22} K \end{bmatrix} \begin{bmatrix} x \\ \bar{x} \\ e \end{bmatrix} + \begin{bmatrix} B_1 \\ LD_{21} \\ LD_{21} \end{bmatrix} w \\ z = \begin{bmatrix} C_1 & D_{12} K & -D_{12} K \end{bmatrix} \begin{bmatrix} x \\ \bar{x} \\ e \end{bmatrix}, \ \forall t \in [t_k, t_{k+1}) \\ e = \bar{x} - \hat{x} = 0, \ t = t_{k+1} \end{array} \tag{2}$$

We will also use the following definitions:

$$\begin{array}{l} \varphi(t) = \begin{bmatrix} x(t) \\ \bar{x}(t) \\ e(t) \end{bmatrix}, \quad \Lambda = \begin{bmatrix} A & B_2 K & -B_2 K \\ LC_2 & \hat{A} - L\hat{C}_2 + \hat{B}_2 K + L\tilde{D}_{22} K & -\hat{B}_2 K - L\tilde{D}_{22} K \\ LC_2 & L\tilde{D}_{22} K - L\hat{C}_2 & \hat{A} - L\tilde{D}_{22} K \end{bmatrix} \\ B_N = \begin{bmatrix} B_1 \\ LD_{21} \\ LD_{21} \end{bmatrix}, \ C_N = \begin{bmatrix} C_1 & D_{12} K & -D_{12} K \end{bmatrix} \end{array} \tag{3}$$

Throughout this paper we will assume that the compensated model is stable and that the transportation delay is negligible. We will assume that the frequency at which the network updates the state in the controller is constant. The goal is to find the smallest frequency at which the network must update the state in the controller, that is, an upper bound for h, the update time. A necessary and sufficient condition for stability of the output feedback MB-NCS is now presented.

Theorem 1. *The non-disturbed output feedback MB-NCS described by (1) is globally exponentially stable around the solution* $\begin{bmatrix} x^T & \bar{x}^T & e^T \end{bmatrix}^T = \mathbf{0}$ *if and only if the eigenvalues of* $M = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} e^{\Lambda h} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *are inside the unit circle.*

A detailed proof for Theorem 1 can be found in [2].

Before defining the performance measures previously described a brief summary of the lifting technique is presented. As it was pointed out, lifting can transform a periodic linear system such as a MB-NCS into a discrete linear time invariant system with operator-valued parameters. These parameters are computed for a class of MB-NCS and used throughout the paper.

2 Continuous Lifting Technique

We will give a brief introduction into the Lifting technique. We need to define two Hilbert spaces, the first space is defined as follows:

$$L_2[0,h) = \left\{ u(t) / \int_0^h u^T(t)u(t)dt < \infty \right\} \tag{4}$$

The second Hilbert space of interest is formed by an infinite sequence of $L_2[0,h)$ spaces and is defined:

$$\begin{aligned} & l_2\left(\mathrm{Z}, L_2[0,h)\right) = l_2 = \\ & = \left\{ [..., u_{-2}, u_{-1}, u_0, u_1, u_2, ...]^T / u_i \in L_2[0,h),\ \sum_{-\infty}^{+\infty} \int_0^h u_j^T(t)u_j(t)dt < \infty \right\} \end{aligned} \tag{5}$$

Now we can define the lifting operator $\mathcal{L}$ as mapping from L_{2e} (L_2 extended) to l_2:

$$\begin{aligned} & \mathcal{L} : L_{2e} \to l_2,\ \mathcal{L}u(t) = [..., u_{-2}, u_{-1}, u_0, u_1, u_2, ...]^T \\ & \text{where } u_k(\tau) = u(\tau + kh),\ \tau \in [0,h) \end{aligned} \tag{6}$$

It can be shown that $\mathcal{L}$ preserves inner products and thus is norm preserving [10]. Since L is surjective, it is an isomorphism of Hilbert spaces. So, lifting basically transforms a continuous function into a discrete function where each element of the sequence is a continuous function restricted to $[0,h)$.

As an example of the application of this lifting technique we will compute the lifted parameters of a MB-NCS with output feedback previously presented (1). This system is clearly h periodic, and therefore we expect to get, after the lifting procedure, an LTI system of the form:

$$\widehat{\varphi}_{k+1} = \widehat{A}\widehat{\varphi}_k + \widehat{B}\widehat{w}_k,\ \ \widehat{z}_k = \widehat{C}\widehat{\varphi}_k + \widehat{D}\widehat{w}_k \tag{7}$$

To obtain the operators we "chop" the time response of the system described in (1) and evaluate at times kh. The lifted parameters can be calculated as:

$$\begin{aligned} \widehat{A} &= \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} e^{\Lambda h}, \ \widehat{B}\widehat{w} = \int_0^h \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} e^{\Lambda(h-s)} B_N \widehat{w}(s)ds \\ \widehat{C} &= C_N e^{\Lambda \tau}, \qquad \widehat{D}\widehat{w} = C_N \int_0^\tau e^{\Lambda(\tau-s)} B_N \widehat{w}(s)ds \end{aligned} \tag{8}$$

The new lifted system is now a LTI discrete system. Note the dimension of the state space is left unchanged, but the input and output spaces are now infinite dimensional. Nevertheless, the new representation [12, 13] allows extending the results available for discrete LTI systems to the lifted domain. These tools have been traditionally used to analyze and synthesize sample and hold devices, and digital controllers. It is to be noted, though, that in this application the discrete part is embedded in the controller that doesn't operate in the same way a typical sampled system does. Here, for instance, the controller gain operates over a continuous signal, as opposed to over a discrete signal as it is customary in sampled data systems.

3 An H2 Norm Characterization of a MB-NCS

It is clear that, since the MB-NCS is h-periodic, there is no transfer function in the normal sense whose H2 norm can be calculated [10]. For LTI systems the H2 norm can be computed by obtaining the 2-norm of the impulse response of the system at $t = t_0$. We will extend this definition to specify an H2 norm, or more properly, to define an H2-like performance index [10]. We will call this performance index Extended H2 Norm. We will study the extended H2 norm of the MB-NCS with output feedback studied in the previous section and shown in Figure 1. The Extended H2 Norm is defined as:

$$\|G_{zw}\|_{xh2} = \left(\sum_i \|G_{zw}\delta(t_0)\, e_i\|_2^2 \right)^{1/2} \tag{9}$$

Theorem 2. *The Extended H2 Norm, $\|G\|_{xh2}$, of the Output Feedback MB-NCS is given by $\|G\|_{xh2} = \left(trace\left(B_N^T X B_N\right)\right)^{1/2}$ where X is the solution of the discrete Lyapunov equation $M(h)^T X M(h) - X + W_o(0,h) = 0$ and $W_o(0,h)$ is the observability Gramian computed as $W_o(0,h) = \int_0^h e^{\Lambda^T t} C_N^T C_N e^{\Lambda t} dt$.*

Proof. We will compute the extended H2 norm of the system by obtaining the 2-norm of the objective signal z to an impulse input $w = \delta(t - t_0)$. It can be shown that the response of the system to an input $w = \delta(t - t_0)$ (assuming that the input dimension is one) is:

$$z(t) = \begin{bmatrix} C_1 \\ D_{12}K \\ -D_{12}K \end{bmatrix} \varphi(t),\ \varphi(t) = e^{\Lambda(t-t_k)} \left(\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} e^{\Lambda h} \right)^k \begin{bmatrix} B_1 & LD_{21} & LD_{21} \end{bmatrix}. \tag{10}$$

With

$$\begin{aligned} \varphi(t) &= \begin{bmatrix} x(t) & \bar{x}(t) & e(t) \end{bmatrix}, \\ \Lambda &= \begin{bmatrix} A & B_2K & -B_2K \\ LC_2 & \hat{A} - L\hat{C}_2 + \hat{B}_2K + L\tilde{D}_{22}K & -\hat{B}_2K - L\tilde{D}_{22}K \\ LC_2 & L\tilde{D}_{22}K - L\hat{C}_2 & \hat{A} - L\tilde{D}_{22}K \end{bmatrix}, \\ h &= t_{k+1} - t_k. \end{aligned}$$

So we can compute the 2-norm of the output:

$$\begin{aligned} \|z\|_2^2 &= \int_{t_0}^{\infty} z(t)^T z(t) dt \\ &= \int_{t_0}^{\infty} B_N^T \left(M(h)^T\right)^k e^{\Lambda^T(t-t_k)} C_N^T C_N e^{\Lambda(t-t_k)} \left(M(h)\right)^k B_N dt \end{aligned}$$

where

$$M(h) = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} e^{\Lambda h},\ B_N = \begin{bmatrix} B_1 & LD_{21} & LD_{21} \end{bmatrix},\ C_N = \begin{bmatrix} C_1 \\ D_{12}K \\ -D_{12}K \end{bmatrix} \tag{11}$$

It is easy to see that the norm of a system with more than one inputs can be obtained by taking the norm of the integral shown in (11). So at this point we can drop our assumption of working with a one dimension input system. We will concentrate now on the integral expression (11).

$$\begin{aligned} \Sigma(h) &= \int_{t_0}^{\infty} B_N^T \left(M(h)^T\right)^k e^{\Lambda^T(t-t_k)} C_N^T C_N e^{\Lambda(t-t_k)} \left(M(h)\right)^k B_N dt \\ &= B_N^T \left(\sum_{i=0}^{\infty} \int_{t_i}^{t_{i+1}} \left(M(h)^T\right)^i e^{\Lambda^T(t-t_i)} C_N^T C_N e^{\Lambda(t-t_i)} \left(M(h)\right)^i dt \right) B_N \\ &= B_N^T \left(\sum_{i=0}^{\infty} \left(M(h)^T\right)^i W_o(0,h) \left(M(h)\right)^i \right) B_N \end{aligned} \tag{12}$$

where

$$W_o(0,h) = \int_0^h e^{\Lambda^T t} C_N^T C_N e^{\Lambda t} dt$$

Note that $W_o(0,h)$ has the form of the observability Gramian. Also note the summation resembles the solution of a discrete Lyapunov equation. This Lyapunov equation can be expressed as:

$$M(h)^T X M(h) - X + W_o(0,h) = 0 \tag{13}$$

In this equation we note that $M(h)$ is a stable matrix if and only if the networked system is stable. Note that $W_o(0,h)$ is a positive semi definite matrix. Under these conditions the solution X will be positive semi definite.$\diamondsuit$

Note that the observability gramian can be factorized as $W_o(0,h) = C_{aux}^T C_{aux} = \int_0^h e^{\Lambda^T t} C_N^T C_N e^{\Lambda t} dt$. This allow us to compute the norm of the system as the norm of an equivalent discrete LTI system.

Corollary 1. *Define* $C_{aux}^T C_{aux} = \int_0^h e^{\Lambda^T t} C_N^T C_N e^{\Lambda t} dt$ *and the auxiliary discrete system* G_{aux} *with parameters:*$A_{aux} = M(h)$, $B_{aux} = B_N, C_{aux}$, *and* $D_{aux} = 0$ *then the following holds:*

$$\|G_{zw}\|_{xh2} = \|G_{aux}\|_2 \tag{14}$$

Example 1. We now present an example using a double integrator as the plant. The plant dynamics are given by: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$; $B_1 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}$; $B_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$; $C_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$; $C_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $D_{12} = 0.1$; $D_{21} = 0.1$; $D_{22} = 0$. We will use the state feedback controller $K = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$. A state estimator with gain $L = \begin{bmatrix} 20 & 100 \end{bmatrix}$ is used to place the state observer eigenvalues at -10. We will use a plant model with the following dynamics: $\hat{A} = \begin{bmatrix} 0.1634 & 0.8957 \\ -0.1072 & -0.1801 \end{bmatrix}$, $\hat{B}_2 = \begin{bmatrix} -0.1686 & 1.0563 \end{bmatrix}$, $\hat{C}_2 = \begin{bmatrix} 0.8764 \\ 0.1375 \end{bmatrix}$, and $\hat{D}_{22} = -0.1304$. This model yields a stable NCS for update times up to approximately 7.5 sec. In Figure 2 we plot the extended H2 norm of the system as a function of the updates times.

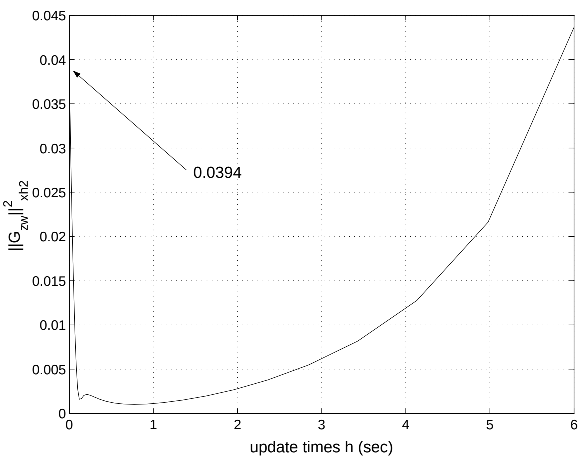

Fig. 2. Extended H2 norm of the system as a function of the update times.

Note that as the update time of the MB-NCS approaches zero, the value of the norm approaches the norm of the non-networked compensated system.

Also note that the optimal update time is around 0.8 sec, and it starts to degrade as the update times become smaller. This pattern is repeated with other norms as shown in the next example.

4 A Generalized H2 Norm for MB-NCS

In the previous section the Extended H2 Norm was introduced to study the performance of MB-NCS, this norm was defined as the norm of the output of the system when a unit impulse at $t = t_0$ is applied to the input. But since the MB-NCS is a time varying system it may seem inappropriate to apply this input only at $t = t_0$. By letting the input be $\delta(t-\tau)$ we arrive to an alternate definition. Since the system is h periodic we only need to consider $\tau \in [t_0, t_0+h)$. We will call this norm the Generalized H2 Norm and we define it as:

$$\|G_{zw}\|_{gh2} = \left(\frac{1}{h} \int_{t_0}^{t_0+h} \left(\sum_i \|G_{zw}\delta(t-\tau)e_i\|_2^2 \right) d\tau \right)^{1/2} \tag{15}$$

A detailed study of this norm can be found in [10]. Note that this norm evaluates the time average of the system response to the impulsive function applied at different times. Another option for a generalized norm could have replaced the time average by the maximum over time. However, as it is shown later, there is relationship between the time averaged norm (15) and the norm of the operator-valued transfer function of G_{zw} that can be useful for frequency domain analysis. We will now show some relations arising from this norm.

Let a continuous-time linear transformation $G : L_2[0,h) \to L_2[0,\infty)$ be defined by:

$$(Gu)(t) = \int_0^t g(t,\tau)\, u(\tau)\, d\tau \tag{16}$$

Where $g(t,\tau)$ is the impulse response of G. Let G be periodic and let its Hilbert-Schmidt norm $\|G\|_{HS}$ be defined as:

$$\|G\|_{HS} = \left(\int_0^h \int_0^\infty trace\left[g(t,\tau)^T g(t,\tau) \right] dt d\tau \right)^{1/2} \tag{17}$$

Then it is clear that:

$$\|G_{zw}\|_{gh2} = \frac{1}{\sqrt{h}} \|G_{zw}\|_{HS} \tag{18}$$

Note the slight abuse of notation since originally G_{zw} was considered a transformation with domain $L_2[0,\infty)$ while the Hilbert-Schmidt norm in (17) is defined for transformations with domain on $L_2[0,h)$. Now denote the lifted operator $\widehat{G}_{zw} = \mathcal{L} G_{zw} \mathcal{L}^{-1}$ with input-output relation given by the convolution:

$$\begin{array}{l} \widehat{z}_k = \sum_{l=0}^{k} \widehat{g}_{k-l} \widehat{w}_l \\ \text{where} \\ \widehat{g}_k : L_2[0,h) \to L_2[0,h) \text{ and } \left(\widehat{g}_k u\right)(t) = \int_0^h g\left(t+kh,\tau\right) u\left(\tau\right) d\tau \end{array} \tag{19}$$

The Hilbert-Schmidt operator for $\widehat{g}_k$ is given by:

$$\left\|\widehat{g}_k\right\|_{HS} = \left(\int_0^h \int_0^h trace\left[g\left(t+kh,\tau\right)^T g\left(t+kh,\tau\right)\right] dt d\tau\right)^{1/2} \tag{20}$$

Then it is easy to show that:

$$\left\|G_{zw}\right\|_{HS}^2 = \sum_{k=0}^{\infty} \left\|\widehat{g}_k\right\|_{HS}^2 = \left\|\widehat{g}\right\|_2^2 \tag{21}$$

The last expression shows a relationship between the discrete lifted representation of the system and the Generalized H2 Norm. Finally we will show the relationship between the Generalized H2 Norm and the norm of an operator-valued transfer function:

$$\tilde{g}\left(\lambda\right) = \sum_{k=0}^{\infty} \widehat{g}_k \lambda^k \tag{22}$$

By defining in a similar way the λ-transform for the input and output of the system we get:

$$\tilde{z}\left(\lambda\right) = \tilde{g}\left(\lambda\right) \tilde{w}\left(\lambda\right) \tag{23}$$

Note that for every λ in their respective regions of convergence, $\tilde{w}\left(\lambda\right)$ and $\tilde{z}\left(\lambda\right)$ are functions on $[0,h)$; while $\tilde{g}\left(\lambda\right)$ is a Hilbert-Schmidt operator. Define the Hardy space $H_2\left(D, HS\right)$ with operator-valued functions that are analytic in the open unit disc, boundary functions on ∂D, and with finite norm:

$$\left\|\tilde{g}\right\|_2 = \left[\frac{1}{2\pi} \int_0^{2\pi} \left\|\tilde{g}\left(e^{j\theta}\right)\right\|_{HS}^2 d\theta\right]^{1/2} \tag{24}$$

Note the norm in $H_2\left(D, HS\right)$ is a generalization of the norm in $H_2\left(D\right)$ by replacing the trace norm by the Hilbert-Schmidt norm. It can be shown that:

$$\left\|G_{zw}\right\|_{HS}^2 = \left\|\widehat{g}\right\|_2^2 = \left\|\tilde{g}\right\|_2^2 \tag{25}$$

We will now show how to calculate the Generalized H2 Norm of the Output Feedback MB-NCS. Define the auxiliary discrete LTI system:

$$G_{aux} \stackrel{S}{=} \left[\begin{array}{c|c} \widehat{A} & B_{aux} \\ \hline C_{aux} & 0 \end{array}\right] \tag{26}$$

Where:

$$\begin{aligned} B_{aux}B_{aux}^T &= \int_0^h \left(\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} e^{\Lambda\tau} B_N B_N^T e^{\Lambda^T\tau} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) d\tau \\ C_{aux}^T C_{aux} &= \int_0^h \left(e^{\Lambda^T\tau} C_N^T C_N e^{\Lambda\tau} \right) d\tau \end{aligned} \tag{27}$$

Theorem 3. *The Generalized H2 Norm,* $\|G_{zw}\|_{gh2}$*, of the Output Feedback MB-NCS is given by* $\|G_{zw}\|_{gh2} = \frac{1}{\sqrt{h}} \left(\left\| \widehat{D} \right\|_{HS}^2 + \|G_{aux}\|_2^2 \right)^{1/2}$.

Proof.
The transfer function for G_{zw} can be written as:

$$\tilde{g}(\lambda) = \widehat{D} + \widehat{C}\left(\tilde{g}_t(\lambda)\right)\widehat{B} \text{ with } \tilde{g}_t(\lambda) = \left[\begin{array}{c|c} \widehat{A} & I \\ \hline I & 0 \end{array}\right] \tag{28}$$

Note that $\tilde{g}_t(\lambda)$ is a matrix-valued function and that $\tilde{g}_t(0) = 0$ therefore the two functions on the right of (28) are orthogonal and:

$$h\|G_{zw}\|_{gh2}^2 = \|\tilde{g}(\lambda)\|_2^2 = \left\|\widehat{D}\right\|_{HS}^2 + \left\|\widehat{C}\left(\tilde{g}_t(\lambda)\right)\widehat{B}\right\|_2^2 \tag{29}$$

The second norm on the right can be calculated as:

$$\left\|\widehat{C}\left(\tilde{g}_t(\lambda)\right)\widehat{B}\right\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} \left\|\widehat{C}\left(\tilde{g}_t\left(e^{j\theta}\right)\right)B\right\|_{HS}^2 d\theta \tag{30}$$

By fixing θ the integrand $F = \widehat{C}\left(\tilde{g}_t\left(e^{j\theta}\right)\right)\widehat{B}$ is a Hilbert-Schmidt operator with impulse response:

$$f(t,\tau) = C_N e^{\Lambda t} \tilde{g}_t\left(e^{j\theta}\right) \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} e^{\Lambda(h-\tau)} B_N \tag{31}$$

Then:

$$\begin{aligned}
\|F\|_{HS}^2 &= trace\left[\int_0^h \int_0^h f(t,\tau)^* f(t,\tau)\, dt d\tau\right] \\
&= trace\left[\int_0^h B_N^T e^{\Lambda^T(h-\tau)} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \tilde{g}_t\left(e^{j\theta}\right)^* C_{aux}^T C_{aux} \tilde{g}_t\left(e^{j\theta}\right) \right. \\
&\qquad \left. \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} e^{\Lambda(h-\tau)} B_N d\tau \right] \\
&= trace\left[\int_0^h \left(\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} e^{\Lambda(h-\tau)} B_N B_N^T e^{\Lambda^T(h-\tau)} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) d\tau \right. \\
&\qquad \left. \left(\tilde{g}_t\left(e^{j\theta}\right)^* C_{aux}^T C_{aux} \tilde{g}_t\left(e^{j\theta}\right) \right) \right] \\
&= trace\left[B_{aux} B_{aux}^T \tilde{g}_t\left(e^{j\theta}\right)^* C_{aux}^T C_{aux} \tilde{g}_t\left(e^{j\theta}\right)\right] \\
&= trace\left[B_{aux}^T \tilde{g}_t\left(e^{j\theta}\right)^* C_{aux}^T C_{aux} \tilde{g}_t\left(e^{j\theta}\right) B_{aux}\right]
\end{aligned} \tag{32}$$

So (30) can be calculated as the H2 norm of $C_{aux}\tilde{g}_t\left(e^{j\theta}\right) B_{aux}$ which corresponds to the H2 norm of G_{aux}.$\diamondsuit$

To calculate the Generalize H2 Norm several calculations need to be done, among these are:

$$\begin{aligned}
&\left\|\widehat{D}\right\|_{HS}^2 = trace\left(\int_0^h \int_0^t B_N^T e^{\Lambda^T \tau} C_N^T C_N e^{\Lambda \tau} B_N d\tau dt\right) \\
&B_{aux} B_{aux}^T = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} P_{22}^T P_{12} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix} = \exp\left(h \begin{bmatrix} -\Lambda & B_N B_N^T \\ 0 & \Lambda^T \end{bmatrix}\right) \\
&C_{aux}^T C_{aux} = M_{22}^T M_{12}; \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix} = \exp\left(h \begin{bmatrix} -\Lambda^T & C_N^T C_N \\ 0 & \Lambda \end{bmatrix}\right)
\end{aligned} \tag{33}$$

Note that in this particular case it was relatively easy to separate the infinite dimensional components of the system from a finite dimensional core component. This is not always possible. In particular one might be tempted to apply the previous techniques to obtain a finite dimensional auxiliary discrete LTI that can be used to solve an H2 optimal control problem. The described separation technique can't be carried out since the controller and observer gains operate over continuous signals. Nevertheless it will be shown later how to address an H2 optimal control problem using other techniques.

Example 2. We now calculate the generalized H2 norm for the same system studied in the previous example. Some computational issues have to be addressed in order to do this. In particular the formulas given in (33) may yield inaccurate results because of scaling issues. In particular for the calculation of B_{aux} and C_{aux} the term (1,1) of the exponentials calculated may be too large in comparison with the other terms, this is because of the negative sign in front of the stable matrix Λ. Direct integration yields better results. Also the

Cholesky factorization is usually only an approximation. The results though seem to represent reality in a reasonable way. This is verified by varying the tolerances in the integration algorithm and by measuring the error in the Cholesky factorization. The calculated generalized H2 norm is shown below for the same range of update times used for the previous example:

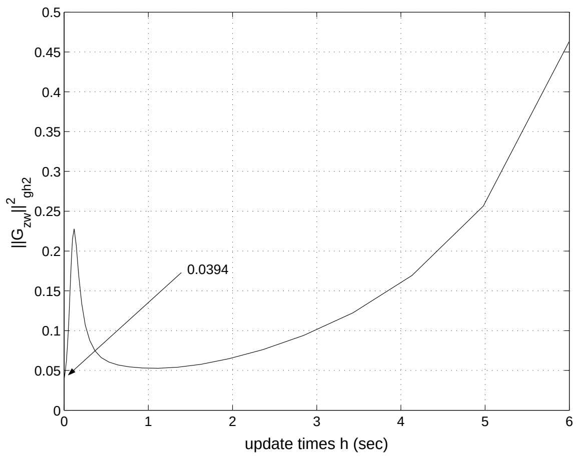

Fig. 3. Generalized H2 norm of the system as a function of the update times.

In this example we see that the norm converges again to the non-networked H2 norm of the system as the update time goes to zero. The optimal update time in this case is around 1 sec, this is somewhat consistent with the previous example where the optimal update time is around 0.8 sec. Both examples coincide in that after the update time of 1 sec the performance starts to degrade. Since both performance measurements are defined in a different manner no real comparison can be made between them. It seems though that the Generalized H2 Norm is more appropriate since it considers the application of the impulse input at different times. Also its link with a well-defined operator-valued transfer function makes it very attractive. The next section presents an alternate parameter representation that overcomes the inconveniencies of dealing with infinite dimensional operators.

5 Optimal Controllers for MB-NCS

In this section we address the issue of designing optimal controllers for MB-NCS. We saw previously that lifting can transform a periodic system such as the MB-NCS into a discrete LTI system. Most results for the design of optimal controllers for discrete systems directly apply to the lifted system. Since the parameters of the lifted system are infinite dimensional, computations using the integral representation given in (8) can be difficult. This is evident when one considers operators such as $\left(I - \widehat{D}^{*}\widehat{D}\right)^{-1}$, which appears for instance in sampled data Hinf problems.

To circumvent some of the problems associated with optimal control problems, an auxiliary discrete LTI system is obtained so that its optimal controller also optimizes the lifted system. The separation of the infinite dimensionality from the problem is not always guaranteed. In particular we note that the controller for the auxiliary system works in the discrete time domain while the controller for the lifted system representing the MB-NCS in (8) works in continuous time. This means the controller has to be obtained using the lifted parameters directly.

In this section we start by giving a brief summary of an alternative representation of the lifted parameters proposed by Mirkin & Palmor [12, 13]. This alternative representation allows performing complex computations using lifted parameters directly. Results on the computation of an optimal sampler, hold, and controller are shown and their equivalence with the components of the output feedback MB-NCS is shown.

The representation of lifted parameters proposed by Mirkin & Palmor considers the lifted parameters as continuous LTI systems operating over a finite time interval. The main advantage of such representation lies in the possibility of simplifying operations over the parameters to algebraic manipulations over LTI systems with two-point boundary conditions. These manipulations can then be performed using well-know state-space machinery.

Consider the following LTI system with two-point boundary conditions (STPBC):

$$\begin{aligned} G : \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \\ \Omega x(0) &+ \Upsilon x(h) = 0 \end{aligned} \tag{34}$$

Here the square matrices Ω and Υ define the boundary conditions. In is said that the boundary conditions are well-posed if $x(t) = 0$ is the only solution to (34) when $u(t) = 0$. It can be verified that the STPBC G has well-posed boundary conditions if and only if the matrix:

$$\Xi_G = \Omega + \Upsilon e^{Ah} \tag{35}$$

is non-singular. If the STPBC G has well-posed boundary conditions, then its response is uniquely determined by the input $u(t)$ and is given as follows:

$$y(t) = Du(t) + \int_0^h K_G(t,s)\,u(s)\,ds \tag{36}$$

where the kernel $K_G(t,s)$ is given by:

$$K_G(t,s) = \begin{cases} Ce^{At}\Xi_G^{-1}\Omega e^{-As}B & \text{if } 0 \leq s \leq t \leq h \\ -Ce^{At}\Xi_G^{-1}\Upsilon e^{A(h-s)}B & \text{if } 0 \leq t \leq s \leq h \end{cases} \tag{37}$$

We will use the following notation to represent (34):

$$G = \left(\frac{A}{C} \boxed{\Omega \rightleftharpoons \Upsilon} \frac{B}{D} \right) \tag{38}$$

The following is a list of manipulations that are used to perform operations over STPBCs.

1) Adjoint System:

$$G^* = \left(\frac{-A^T}{-B^T} \boxed{e^{A^T h}\Upsilon^T \Xi_G^{-T} \rightleftharpoons \Omega^T \Xi_G^{-T} e^{A^T h}} \frac{C^T}{D^T} \right) \tag{39}$$

2) Similarity Transformation: (for T and S non singular)

$$TGT^{-1} = \left(\frac{TAT^{-1}}{CT^{-1}} \boxed{S\Omega T^{-1} \rightleftharpoons S\Upsilon T^{-1}} \frac{TB}{D} \right) \tag{40}$$

3) Addition:

$$G_1 + G_2 = \left(\frac{\begin{matrix} A_1 & 0 \\ 0 & A_2 \end{matrix}}{\begin{matrix} C_1 & C_2 \end{matrix}} \boxed{\begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{bmatrix} \rightleftharpoons \begin{bmatrix} \Upsilon_1 & 0 \\ 0 & \Upsilon_2 \end{bmatrix}} \frac{\begin{matrix} B_1 \\ B_2 \end{matrix}}{D_1 + D_2} \right) \tag{41}$$

4) Multiplication:

$$G_1 G_2 = \left(\frac{\begin{matrix} A_1 & B_1 C_2 \\ 0 & A_2 \end{matrix}}{\begin{matrix} C_1 & D_1 C_2 \end{matrix}} \boxed{\begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{bmatrix} \rightleftharpoons \begin{bmatrix} \Upsilon_1 & 0 \\ 0 & \Upsilon_2 \end{bmatrix}} \frac{\begin{matrix} B_1 D_2 \\ B_2 \end{matrix}}{D_1 D_2} \right) \tag{42}$$

5) Inversion (exists if and only if $\det(D) \neq 0$ and $\det\left(\Omega + \Upsilon e^{\left(A - BD^{-1}C\right)h}\right) \neq 0$)

$$G^{-1} = \left(\frac{A - BD^{-1}C}{-D^{-1}C} \boxed{\Omega \rightleftharpoons \Upsilon} \frac{BD^{-1}}{D^{-1}} \right), \tag{43}$$

This representation reduces the complexity of computing operators such as $\left(I - \widehat{D}^* \widehat{D}\right)^{-1}$. Using the integral representation of (8) one can get that $\xi = \left(I - \widehat{D}^* \widehat{D}\right)^{-1} \omega$ if and only if:

$$\omega(t) = \xi(t) + \int_t^h B_N^T e^{-A^T(t-s)} C_N^T C_N \int_0^s e^{A(s-\tau)} B_N \xi(\tau)\, d\tau ds \tag{44}$$

It is not clear how to solve this equation. On the other hand using the alternative representation we note that:

$$\widehat{D} = \left(\frac{\Lambda}{C_N} \boxed{I \rightleftharpoons 0} \frac{B_N}{0} \right) \tag{45}$$

Using the properties previously listed we obtain:

$$\begin{aligned} \left(I - \widehat{D}^* \widehat{D} \right)^{-1} &= \left(I - \left(\frac{\Lambda}{C_N} \boxed{I \rightleftharpoons 0} \frac{B_N}{0} \right)^* \left(\frac{\Lambda}{C_N} \boxed{I \rightleftharpoons 0} \frac{B_N}{0} \right) \right)^{-1} \\ &= \left(\frac{\begin{matrix} -\Lambda^T & C_N^T C_N \\ -B_N B_N^T & \Lambda \end{matrix}}{-B_N^T \; 0} \boxed{\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \rightleftharpoons \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}} \frac{\begin{matrix} 0 \\ B_N \end{matrix}}{I} \right) \end{aligned} \tag{46}$$

To be able to represent operators with finite dimension domains or ranges such as $\widehat{B}$ and $\widehat{C}$ two new operators are defined. Given a number $\theta \in [0, h]$, the impulse operator $\mathcal{I}_\theta$ transforms a vector $\eta \in \mathrm{R}^n$ into a modulated impulse as follows:

$$\varsigma = \mathcal{I}_\theta \eta \quad \Leftrightarrow \quad \varsigma(t) = \delta(t - \theta)\, \eta \tag{47}$$

Also define the sample operator $\mathcal{I}_\theta^*$, which transforms a continuous function $\varsigma \in \mathrm{C}_n [0, h]$ into a vector $\eta \in \mathrm{R}^n$ as follows:

$$\eta = \mathcal{I}_\theta^* \varsigma \quad \Leftrightarrow \quad \eta = \varsigma(\theta) \tag{48}$$

Note that the representation of $\mathcal{I}_\theta^*$ is as the adjoint of $\mathcal{I}_\theta$, even when this is not strictly true, it is easy to see that given an $h \geq \theta$ the following equality holds:

$$\langle \varsigma, \mathcal{I}_\theta \eta \rangle = \int_0^h \varsigma(\tau)^T (\mathcal{I}_\theta \eta)(\tau)\, d\tau = \varsigma(\theta)^T \eta = \langle \mathcal{I}_\theta^* \varsigma, \eta \rangle \tag{49}$$

The presented results allow to make effective use of the impulse and sample operator. Namely the last two lemmas show how to absorb the impulse operators into an STPBC. Now let us present a result that links the solutions of the lifted algebraic discrete Riccati equation and the algebraic continuous Riccati equation for the continuous system for the H2 control problem.

Lemma 1. *Let the lifted algebraic discrete Riccati equation for the lifted system $\widehat{G} = LGL^{-1}$ be as follows:*

$$\begin{aligned} &\widehat{A}^T \widehat{X} \widehat{A} - \widehat{X} + \widehat{C}^* \widehat{C} \\ &\quad - \left(\widehat{A}^T \widehat{X} \widehat{B} + \widehat{C}^* \widehat{D} \right) \left(\widehat{D}^* \widehat{D} + \widehat{B}^T \widehat{X} \widehat{B} \right)^{-1} \left(\widehat{D}^* \widehat{C} + \widehat{B}^T \widehat{X} \widehat{A} \right) = 0 \end{aligned} \tag{50}$$

and let the algebraic continuous Riccati equation for G be:

$$A^T X + XA + C^T C - \left(XB + C^T D\right)\left(D^T D\right)^{-1}\left(D^T C + B^T X\right) = 0 \quad (51)$$

then the conditions for existence of a unique stable solution for both Ricatti equations are equivalent, moreover if they exist, then $\widehat{X} = X$.

This implies that in order to solve the optimal control problem we just need to solve the regular continuous Ricatti equation. We can for example obtain the optimal H2 state feedback "gain" given by

$$\widehat{F} = -\left(\widehat{D}^*\widehat{D} + \widehat{B}^* X\widehat{B}\right)^{-1}\left(\widehat{D}^*\widehat{C} + \widehat{B}^* X\widehat{A}\right).$$

It can be shown [12, 13] that:

$$\widehat{F} = \left(\frac{A+BF}{F}\boxed{I \rightleftharpoons 0}\frac{I}{0}\right) \quad (52)$$

Here F is the H2 optimal control gain for the continuous system. Note that the expression in (52) exactly represents the dynamics of the actuator/controller for the state feedback MB-NCS when the modelling errors are zero and the feedback gain is the H2 optimal feedback gain. Finally, we present next a result that obtains the H2 optimal sampler, hold and controller.

Lemma 2. *Given the standard assumptions, when the hold device is given by $(Hu)(kh+\tau) = \phi_H(\tau)\, u_k$, $\forall \tau \in [0,h]$ and the sample device is given by $(Sy)_k = \int_0^h \phi_S(\tau)\, y(kh-\tau)\, d\tau$, the H2 optimal hold, sampler, and discrete controller for the lifted system $\widehat{G} = LGL^{-1}$ with*

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$

are as follows:

$$\begin{array}{ll} \textit{Hold:} & \phi_H(\tau) = Fe^{(A+B_2F)\tau} \\ \textit{Sampler:} & \phi_S(\tau) = -e^{(A+LC_2)\tau}L \\ \textit{Controller:} & K_d = \left[\begin{array}{c|c} \Theta & I \\ \hline I & 0 \end{array}\right] \\ \textit{where:} & \Theta = e^{(A+B_2F)h} + \int_0^h e^{(A+LC_2)(h-\tau)}LC_2 e^{(A+B_2F)\tau}d\tau \end{array} \quad (53)$$

Remarks: Note that the H2 optimization problem solved in [13] is related to the Generalized H2 norm previously presented. That is, replacing the trace norm with the Hilbert-Schmidt norm.

As it has been observed, there is a strong connection between the H2 optimal hold of a sampled system and the H2 optimal controller of the non-sampled system. As pointed out in [13] it is clear that the H2 optimal hold attempts to recreate the optimal control signal that would have been generated by the H2 optimal controller in the non-sampled case. That is, the H2 optimal hold calculated in [13] generates a control signal identical to the one generated by the non-sampled H2 optimal controller in the absence of noise and disturbances.

Another connection exists between the H2 optimal sampler, hold, and discrete controller calculated in [13] and the output feedback MB-NCS. It is clear that when the modeling errors are zero and the gain is the optimal H2 gain, the optimal hold has the same dynamics as the controller/ actuator in the output feedback MB-NCS. The same equivalence can be shown between the combination of optimal sampler/discrete controller dynamics and the output feedback MB-NCS observer.

The techniques shown here can be used to solve robust optimal control problems that consider the modeling error. This is possible due to the alternative representation that allows the extension of traditional optimal control synthesis techniques to be used with the infinite dimensional parameters that appear in the lifted domain.

6 Conclusions

The study of the performance of MB-NCS shows that a large portion of the available literature on sampled data systems cannot be directly applied to MB-NCS. Moreover, different definitions of performance yield different performance curves. For a constant controller and observer gains it was shown that the best transmission times are not necessarily the smallest ones. Using an alternate representation of the lifted parameters, a connection between the optimal hold, sampler, and discrete controller and the output feedback MB-NCS was established. This representation opens a large new area of research in robust MB-NCS.

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