Weak-Interactions in Atoms

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1 Elecrto-Weak Coupling

The weak coupling of bound atomic electrons to the nucleus through the exchange of Z_0 bosons is primarily responsible for parity nonconservation (PNC) in atoms. The PNC part of the electron-nucleus interaction Hamiltonian splits into two parts, $H^{(1)} = A_e V_N$ from the product of axial-vector electron A_e and vector nucleon V_N currents, and $H^{(2)}$ from the product of vector electron V_e and axial-vector nucleon A_N currents. These contributions are given in terms of electron and nucleon field operators as [1]

$$H^{(1)} = \frac{G}{\sqrt{2}} \left(\bar{\psi}_e \gamma_\mu \gamma_5 \psi_e\right) \sum_i \left[c_{1p} \left(\bar{\psi}_{pi} \gamma^\mu \psi_{pi}\right) + c_{1n} \left(\bar{\psi}_{ni} \gamma^\mu \psi_{ni}\right)\right], \quad (1)$$

$$H^{(2)} = \frac{G}{\sqrt{2}} \left(\bar{\psi}_e \gamma_\mu \psi_e\right) \sum_i \left[c_{2p} \left(\bar{\psi}_{pi} \gamma^\mu \gamma_5 \psi_{pi}\right) + c_{2n} \left(\bar{\psi}_{ni} \gamma^\mu \gamma_5 \psi_{ni}\right)\right], \quad (2)$$

where the Standard-model coupling constants are

$$c_{1p} = \tfrac{1}{2}\left(1 - 4\sin^2\theta_W\right) \approx 0.038, \quad (3)$$

$$c_{1n} = -\tfrac{1}{2}, \quad (4)$$

$$c_{2p} = \tfrac{1}{2}\, g_A \left(1 - 4\sin^2\theta_W\right) \approx 0.047, \quad (5)$$

$$c_{2n} = -\tfrac{1}{2}\, g_A \left(1 - 4\sin^2\theta_W\right) \approx -0.047. \quad (6)$$

In the above, $g_A \approx 1.25$ is a scale factor for the partially conserved axial current A_N taken from p. 173 of Ref. [1]. The presently accepted value of Weinberg's angle is $\sin^2\theta_W = 0.23124(24)$.

1.1 Nonrelativistic Reduction

1.1.1 Reduction of $H^{(1)}$

We assume that the nucleons are nonrelativistic and replace the nucleon vector currents in Eq. (1) by

$$\left(\bar{\psi}_p \gamma^\mu \psi_p\right) \to \phi_p^\dagger \phi_p \, \delta_{\mu 0} \qquad \text{and} \qquad \left(\bar{\psi}_n \gamma^\mu \psi_n\right) \to \phi_n^\dagger \phi_n \, \delta_{\mu 0},$$

where ϕ_p and ϕ_n are nonrelativistic field operators. From this we extract an "effective" Hamiltonian to be used in the electron sector, namely

$$H_{\text{eff}}^{(1)} = \frac{G}{2\sqrt{2}} \gamma_5 \left[2\, Z\, c_{1p}\, \rho_p(r) + 2\, N\, c_{1n}\, \rho_n(r) \right]. \tag{7}$$

In this expression, $\rho_p(r)$ and $\rho_n(r)$ proton and neutron density functions normalized to 1, and Z and N are proton and neutron numbers of the nucleus. Assuming $\rho_p(r) = \rho_n(r) = \rho(r)$, we may rewrite the effective Hamiltonian as

$$H_{\text{eff}}^{(1)} = \frac{G}{2\sqrt{2}} \gamma_5 \, Q_w \, \rho(r), \tag{8}$$

where we have introduced the weak charge Q_w defined by

$$Q_w = \left[2\, Z\, c_{1p} + 2\, N\, c_{1n} \right] = -N + Z \left(1 - 4 \sin^2 \theta_W\right).$$

The Dirac matrix γ_5 in the effective Hamiltonian (8) is

$$\gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

1.1.2 Reduction of $H^{(2)}$

The nonrelativistic approximation for the nucleon axial currents in Eq. (2) is

$$\left(\bar{\psi}_p \gamma^\mu \gamma_5 \psi_p\right) \to \phi_p^\dagger \sigma_i \phi_p \, \delta_{\mu i} \qquad \text{and} \qquad \left(\bar{\psi}_n \gamma^\mu \psi_n\right) \to \phi_n^\dagger \sigma_i \phi_n \, \delta_{\mu i}.$$

The corresponding effective Hamiltonian in the electron sector is obtained from

$$H_{\text{eff}}^{(2)} = -\frac{G}{\sqrt{2}} \boldsymbol{\alpha} \cdot \left[c_{2p} \left\langle \phi_p^\dagger \boldsymbol{\sigma} \phi_p \right\rangle + c_{2n} \left\langle \phi_n^\dagger \boldsymbol{\sigma} \phi_n \right\rangle \right], \tag{9}$$

where $\langle \cdots \rangle$ notation designates nuclear matrix elements. Typically, only a few unpaired valence nucleons contribute to this interaction, so the size of the $H^{(2)}$ contribution is smaller than that from $H^{(1)}$ by a factor of $\approx 1/Z$.

Let us examine the angular part of the nuclear matrix element $\left\langle \phi^\dagger \boldsymbol{\sigma} \phi \right\rangle$ for the case of a single nucleon outside closed shells. We can write

$$\langle IM'|\sigma_\mu|IM\rangle = (-1)^{I+L+1/2}\sqrt{6}\,[I] \left\{ \begin{array}{ccc} I & I & 1 \\ 1/2 & 1/2 & L \end{array} \right\} - \begin{array}{|l} {}^{IM'} \\ \quad 1\mu \\ {}_{IM} \end{array} \; . \tag{10}$$

It is also true that

$$\langle IM'|I_\mu|IM\rangle = \sqrt{I(I+1)(2I+1)} - \begin{array}{|l} {}^{IM'} \\ \quad 1\mu \\ {}_{IM} \end{array} \; . \tag{11}$$

We can therefore replace matrix elements of σ_μ by suitable scaled matrix elements of I_μ. Specifically,

$$\sigma_\mu \rightarrow \frac{\langle I\|\sigma\|I\rangle}{\langle I\|I\|I\rangle}\, I_\mu = \frac{\langle I\|\sigma\|I\rangle}{\sqrt{I(I+1)(2I+1)}}\, I_\mu . \tag{12}$$

For the single valence nucleon case,

$$\begin{aligned}
\frac{\langle I\|\sigma\|I\rangle}{\langle I\|I\|I\rangle} &= \sqrt{\frac{6(2I+1)}{I(I+1)}}(-1)^{I+L+1/2} \left\{ \begin{array}{ccc} I & I & 1 \\ 1/2 & 1/2 & L \end{array} \right\} && (13) \\
&= -\frac{I+1}{I(I+1)} \quad \text{for } L = I - 1/2 && (14) \\
&= \frac{I}{I(I+1)} \quad \text{for } L = I + 1/2 && (15) \\
&\equiv \frac{\kappa - 1/2}{I(I+1)}, && (16)
\end{aligned}$$

where $\kappa = \mp(I + 1/2)$ for $I = L \pm 1/2$. Combining this with Eq.(9), we obtain for case of a nucleus with one valence nucleon:

$$H^{(2)}_{\text{eff}} = -\frac{G}{\sqrt{2}}\,\frac{\kappa - 1/2}{I(I+1)}\,\boldsymbol{\alpha}\cdot\mathbf{I}\;\left[c_{2p}\,\rho_{pv}(r) + c_{2n}\,\rho_{nv}(r)\right], \tag{17}$$

where $\rho_{pv}(r)$ or $\rho_{nv}(r)$ are the valence nucleon density functions. In our previous notation, we used $K_2 = c_{2p}$ for a valence proton or $K_2 = c_{2n}$ for a valence neutron and let $\rho_v(r)$ be the associated density, then

$$H^{(2)}_{\text{eff}} = -\frac{G}{\sqrt{2}}\,K_2\,\frac{\kappa - 1/2}{I(I+1)}\,\boldsymbol{\alpha}\cdot\mathbf{I}\,\rho_v(r). \tag{18}$$

1.1.3 Anapole

The electromagnetic interaction of the nuclear anapole moment and the electron may be written

$$H_{\text{eff}}^{(a)} = \frac{G}{\sqrt{2}} K_a \frac{\kappa}{I(I+1)} \boldsymbol{\alpha} \cdot \mathbf{I} \, \rho_v(r). \tag{19}$$

for a nucleus with a single valence nucleon, according to Ref. [2]. It is convenient to combine the two terms that depend on nuclear spin into a single interaction

$$H_{\text{eff}}^{(2')} = \frac{G}{\sqrt{2}} K \frac{\kappa}{I(I+1)} \boldsymbol{\alpha} \cdot \mathbf{I} \, \rho_v(r), \tag{20}$$

where

$$K = K_a - (\kappa - 1/2)/\kappa \, K_2 \, .$$

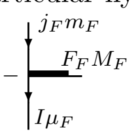
2 Dipole Matrix Element

The weak interaction induces parity violation in atomic states. As a consequence, electric dipole transitions between states of the same parity, normally forbidden, become allowed. If $|I\rangle$ and $|F\rangle$ represent two atomic states of the same nominal parity, then to lowest nonvanishing order, the electric dipole transition matrix element is

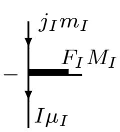
$$\langle F|ez|I\rangle = \sum_n \frac{\langle F|ez|n\rangle\langle n|H_W|I\rangle}{E_n - E_I} + \sum_n \frac{\langle F|H_W|n\rangle\langle n|ez|I\rangle}{E_n - E_F}, \tag{21}$$

where $H_W = H_{\text{eff}}^{(1)} + H_{\text{eff}}^{(2)} + H_{\text{eff}}^{(a)}$ is the effective weak-interaction Hamiltonian discussed above.

Now let us concentrate on particular hyperfine states

$|F_F M_F\rangle \;=\; -$ 

$|j_F m_F\rangle |I \mu_F\rangle$

$|F_I M_I\rangle \;=\; -$ 

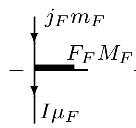
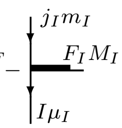
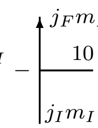
$|j_I m_I\rangle |I \mu_I\rangle$

$$|n\rangle = |n\, j_n m_n\rangle |I \mu_n\rangle$$

Matrix elements of the spin-independent and spin-dependent terms behave differently. Let us consider them in turn

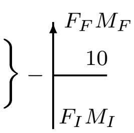
2.1 Spin-independent term

If we consider only the part of the weak interaction $H^{(1)}_{\rm eff}$ that is independent of nuclear spin, then we may write

$$\langle F|ez|I\rangle =$$    $$\delta_{\mu_I\mu_F} \times$$

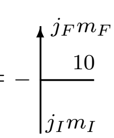
$$\sum_n \left\{ \left. \frac{\langle j_F\|ez\|n\, j_n\rangle\langle n\, j_n\|H^{(1)}\|j_I\rangle}{E_n - E_I} \right|_{\pi_n=-\pi_I} + \left. \frac{\langle j_F\|H^{(1)}\|n\, j_n\rangle\langle n\, j_n\|ez\|j_I\rangle}{E_n - E_I} \right|_{\pi_n=-\pi_F} \right\}, \tag{22}$$

where we have dropped the subscript "eff". Summing over magnetic quantum numbers, this term becomes

$$\langle F|ez|I\rangle = (-1)^{j_F+F_I+I+1}\sqrt{[F_I][F_F]} \left\{ \begin{array}{ccc} F_F & F_I & 1 \\ j_I & j_F & I \end{array} \right\}$$  $$\times$$

$$\sum_{nj_n} \left\{ \left. \frac{\langle j_F\|ez\|n\, j_n\rangle\langle n\, j_n\|H^{(1)}\|j_I\rangle}{E_n - E_I} \right|_{\pi_n=-\pi_I} + \left. \frac{\langle j_F\|H^{(1)}\|n\, j_n\rangle\langle n\, j_n\|ez\|j_I\rangle}{E_n - E_I} \right|_{\pi_n=-\pi_F} \right\}. \tag{23}$$

If we ignore nuclear spin altogether, then we may write

$$\langle j_F\, m_F|ez|j_I\, m_I\rangle =$$  $$\sum_{nj_n}\{\cdots\}\,, \tag{24}$$

where the sum over n is identical to that in Eq. (23). For alkali-metal atoms $j_F = j_I = 1/2$ and it is conventional to define the spin-independent PNC matrix element as

$$E^{(1)}_{\rm PNC} = \langle j_F\, \frac{1}{2}\,|\, ez\,|\, j_I\, \frac{1}{2}\rangle = \frac{1}{\sqrt{6}}\sum_{nj_n}\{\cdots\}$$

Therefore, we may rewrite Eq. (23) in terms of the conventional PNC matrix element for *alkali-metal atoms* as

$$\langle F|ez|I\rangle = (-1)^{F_I+I+3/2}\sqrt{6[F_I][F_F]}\left\{\begin{array}{ccc} F_F & F_I & 1 \\ 1/2 & 1/2 & I \end{array}\right\} E^{(1)}_{\rm PNC} \; - \begin{array}{|l} F_F M_F \\ \hline \quad 10 \\ F_I M_I \end{array} \, . \qquad (25)$$

This expression can be used to extract the spin-independent matrix element $E^{(1)}_{\rm PNC}$ from measurements on individual hyperfine lines and provides a working definition for the experimental PNC matrix element.

2.2 Spin-dependent interaction

Now, let us examine the part of the interaction that depends on nuclear spin, $H^{(2)}_{\rm eff} + H^{(a)}_{\rm eff} \equiv \sum_\mu (-1)^\mu I_{-\mu} H^{(2')}_\mu$. The dipole matrix element may be written:

$$\langle F|ez|I\rangle = - \begin{array}{|l} j_F m_F \\ \hline \quad F_F M_F \\ I \mu_F \end{array} - \begin{array}{|l} j_I m_I \\ \hline \quad F_I M_I \\ I \mu_I \end{array} \sum_\mu (-1)^\mu \langle I\mu_F | I_{-\mu} | I \mu_I\rangle \times$$

$$\sum_{n j_n m_n} \left\{ \frac{\langle j_F m_F|ez|n\, j_n m_n\rangle\langle n\, j_n m_n|H^{(2')}_\mu|j_I m_I\rangle}{E_n - E_I} + \frac{\langle j_F m_F|H^{(2')}_\mu|n\, j_n m_n\rangle\langle n\, j_n m_n|ez|j_I m_I\rangle}{E_n - E_F} \right\} . \qquad (26)$$

We use the fact that

$$\sum_\mu (-1)^\mu \langle I\mu_F|I_{-\mu}|I\mu_I\rangle = \sqrt{I(I+1)[I]} \; - \begin{array}{|l} I\mu_I \\ \hline \quad 1\mu \\ I\mu_F \end{array} \, , \qquad (27)$$

to write

$$\langle F|ez|I\rangle = \sqrt{I(I+1)[I][F_I][F_F]} \; - \begin{array}{|l} j_F m_F \\ \hline \quad F_F M_F \\ I\mu_F \end{array} - \begin{array}{|l} j_I m_I \\ \hline \quad F_I M_I \\ I\mu_I \end{array} - \begin{array}{|l} I\mu_I \\ \hline \quad 1\mu \\ I\mu_F \end{array} \times$$

$$\sum_{n j_n} \left\{ - \begin{array}{|l} j_F m_F \\ \hline \quad 10 \\ j_n m_n \end{array} - \begin{array}{|l} j_n m_n \\ \hline \quad 1\mu \\ j_I m_I \end{array} \frac{\langle j_F\|ez\|n\, j_n\rangle\langle n\, j_n\|H^{(2')}\|j_I\rangle}{E_n - E_I} \right.$$

$$\left. + - \begin{array}{|l} j_F m_F \\ \hline \quad 1\mu \\ j_n m_n \end{array} - \begin{array}{|l} j_n m_n \\ \hline \quad 10 \\ j_I m_I \end{array} \frac{\langle j_F\|H^{(2')}\|n\, j_n\rangle\langle n\, j_n\|ez\|j_I\rangle}{E_n - E_F} \right\} . \qquad (28)$$

After summing over magnetic quantum numbers, this expression reduces to

$$
\langle F|ez|I\rangle = \sqrt{I(I+1)}\sqrt{[I][F_I][F_F]}\; - \begin{array}{l} F_F M_F \\ \quad 10 \\ F_I M_I \end{array} \times
$$
$$
\sum_{nj_n}\Bigg[(-1)^{j_I-j_F+1}\left\{\begin{array}{ccc} F_F & F_I & 1 \\ j_n & j_F & I \end{array}\right\}\left\{\begin{array}{ccc} I & I & 1 \\ j_n & j_I & F_I \end{array}\right\}\frac{\langle j_F\|ez\|n\,j_n\rangle\langle n\,j_n\|H^{(2')}\|j_I\rangle}{E_n-E_I}
$$
$$
+(-1)^{F_I-F_F+1}\left\{\begin{array}{ccc} F_F & F_I & 1 \\ j_n & j_I & I \end{array}\right\}\left\{\begin{array}{ccc} I & I & 1 \\ j_n & j_F & F_F \end{array}\right\}\frac{\langle j_F\|H^{(2')}\|n\,j_n\rangle\langle n\,j_n\|ez\|j_I\rangle}{E_n-E_F}\Bigg]. \tag{29}
$$

For the case of alkali-metal atoms, this expression can be used together with Eq. (25) to *define* a spin-dependent PNC matrix element

$$
E_{\rm PNC} = E^{(1)}_{\rm PNC} + E^{(2)}_{\rm PNC}, \tag{30}
$$

where $E^{(1)}_{\rm PNC}$ was given in the previous subsection:

$$
E^{(1)}_{\rm PNC} = \frac{1}{\sqrt{6}}\sum_{nj_n}\left\{ \left.\frac{\langle j_F\|ez\|n\,j_n\rangle\langle n\,j_n\|H^{(1)}\|j_I\rangle}{E_n-E_I}\right|_{\pi_n=-\pi_I} + \left.\frac{\langle j_F\|H^{(1)}\|n\,j_n\rangle\langle n\,j_n\|ez\|j_I\rangle}{E_n-E_I}\right|_{\pi_n=-\pi_F}\right\}, \tag{31}
$$

and

$$
E^{(2)}_{\rm PNC} = \sqrt{\frac{I(I+1)\,[I]}{6}}(-1)^{F_I+I+3/2}\left\{\begin{array}{ccc} F_F & F_I & 1 \\ 1/2 & 1/2 & I \end{array}\right\}^{-1}
$$
$$
\sum_{nj_n}\Bigg[(-1)^{1}\left\{\begin{array}{ccc} F_F & F_I & 1 \\ j_n & 1/2 & I \end{array}\right\}\left\{\begin{array}{ccc} I & I & 1 \\ j_n & 1/2 & F_I \end{array}\right\}\frac{\langle j_F\|ez\|n\,j_n\rangle\langle n\,j_n\|H^{(2')}\|j_I\rangle}{E_n-E_I}
$$
$$
+(-1)^{F_I-F_F+1}\left\{\begin{array}{ccc} F_F & F_I & 1 \\ j_n & 1/2 & I \end{array}\right\}\left\{\begin{array}{ccc} I & I & 1 \\ j_n & 1/2 & F_F \end{array}\right\}\frac{\langle j_F\|H^{(2')}\|n\,j_n\rangle\langle n\,j_n\|ez\|j_I\rangle}{E_n-E_F}\Bigg]. \tag{32}
$$

3 Reduced Matrix Elements

ez: The reduced matrix element of the dipole matrix element is

$$\langle 2\|ez\|1\rangle = e\langle\kappa_2\|C_1\|\kappa_1\rangle \int_0^\infty dr\, r\,(G_1G_2 + F_2F_1). \tag{33}$$

$H^{(1)}$: Introducing the scale factor

$$\mathcal{F}^{(1)} = \frac{G}{2\sqrt{2}}\, Q_W$$

we find

$$\langle 2|H^{(1)}|1\rangle = i\,\mathcal{F}^{(1)}\delta_{\kappa_2\,-\kappa_1}\delta_{m_2\,m_1} \int_0^\infty dr\,(F_2G_1 - G_2F_1). \tag{34}$$

In Eq. (22) and subsequently, the reduced matrix element of $H^{(1)}$ is defined as the coefficient of the $\delta_{j_2\,j_1}\,\delta_{m_2\,m_1}$. Although this is a unconventional definition, we will use it here. It follows that

$$\langle 2\|H^{(1)}\|1\rangle = i\,\mathcal{F}^{(1)} \int_0^\infty dr\,(F_2G_1 - G_2F_1). \tag{35}$$

$H^{(2')}$: Let us introduce the scale factor

$$\mathcal{F}^{(2')} = \frac{G}{\sqrt{2}}K\,\frac{\kappa}{I(I+1)}$$

and write

$$\langle 2|H^{(2')}_\mu|1\rangle = i\,\mathcal{F}^{(2')} \int_0^\infty dr\rho_v(r)\,\Big[\langle -\kappa_2 m_2|\sigma_\mu|\kappa_1 m_1\rangle F_2G_1 \\ -\langle\kappa_2 m_2|\sigma_\mu|-\kappa_1 m_1\rangle\, G_2F_1\Big]. \tag{36}$$

From this, it follows

$$\langle 2\|H^{(2')}\|1\rangle = i\,\mathcal{F}^{(2')} \int_0^\infty dr\rho_v(r)\,\Big[\langle -\kappa_2\|\sigma\|\kappa_1\rangle F_2G_1 \\ -\langle\kappa_2\|\sigma\|-\kappa_1\rangle\, G_2F_1\Big]. \tag{37}$$

The reduced matrix elements of σ are given by:

$$\langle -\kappa_2\|\sigma\|\kappa_1\rangle = (-1)^{j_2+\bar{l}_2-1/2}\sqrt{6\,[j_1][j_2]}\,\delta_{\bar{l}_2\,l_1}\left\{\begin{array}{ccc} j_1 & j_2 & 1 \\ 1/2 & 1/2 & \bar{l}_2 \end{array}\right\} \tag{38}$$

$$\langle\kappa_2\|\sigma\|-\kappa_1\rangle = (-1)^{j_2+l_2-1/2}\sqrt{6\,[j_1][j_2]}\,\delta_{l_2\,\bar{l}_1}\left\{\begin{array}{ccc} j_1 & j_2 & 1 \\ 1/2 & 1/2 & l_2 \end{array}\right\}. \tag{39}$$

4 Units:

The weak interaction coupling constant G has the value

$$\begin{aligned} G &= 89.61971 \quad \text{eV fm}^3 \\ &= 3.293465 \quad \text{a.u. fm}^3. \end{aligned} \tag{40}$$

The normalized nuclear density function can be written

$$\rho(r) = \frac{3}{4\pi} \frac{1}{\mathcal{N}\, c^3} \frac{1}{\left[1 + \exp\left(\dfrac{r-c}{a}\right)\right]}, \tag{41}$$

where $t = (4\log 3)a$ is the 10%–90% fall-off distance and

$$\mathcal{N} \equiv \mathcal{N}\left(\frac{c}{a}\right)$$

is a normalization factor given by

$$\mathcal{N}(x) = 1 + \frac{\pi^2}{x^2} + \frac{6}{x^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} e^{-nx} \tag{42}$$

Note: $\mathcal{N}(x) \to 1$ as $x \to \infty$. We express c in fm and find that

$$\frac{G}{2\sqrt{2}}\,\rho(r) = \frac{0.277984(2)}{\mathcal{N}\, c_{\text{fm}}^3} \frac{1}{\left[1 + \exp\left(\dfrac{r-c}{a}\right)\right]}, \quad \text{a.u. (energy)} \tag{43}$$

where c_{fm} is the nuclear radius c in fm.

References

[1] E. D. Commins and P. H. Bucksbaum, *Weak interactions of leptons and quarks*, (Cambridge University Press, Cambridge, 1983), p. 343.

[2]