Approximate Coulomb Scattering Amplitudes

The relativistic scattering amplitudes are

$$
\begin{aligned}
f(\theta) &= \frac{1}{2ip}\sum_{l=0}^{\infty}\left[(l+1)\left(e^{2i\delta_{-l-1}}-1\right)+l\left(e^{2i\delta_l}-1\right)\right]P_l(\cos\theta) \\
g(\theta) &= \frac{1}{2ip}\sum_{l=0}^{\infty}\left(e^{2i\delta_{-l-1}}-e^{2i\delta_l}\right)P_l^1(\theta)\ ,
\end{aligned}
$$

where the relativistic Coulomb phase shifts δ_κ are given by

$$\delta_\kappa = \arg\Gamma(\gamma - i\nu) + \eta - \frac{1}{2}\pi\gamma + \frac{1}{2}\pi(l+1)\ ,$$

with (units $\hbar = 1,\ c = 1$),

$$
\begin{aligned}
e^{2i\eta} &= \frac{-\kappa + i\nu'}{\gamma + i\nu} \\
\nu &= \frac{E}{p}\alpha Z \\
\nu' &= \frac{m}{p}\alpha Z \\
\gamma &= \sqrt{\kappa^2 - \alpha^2 Z^2}\ .
\end{aligned}
$$

Introduce the partial wave amplitudes:

$$
\begin{aligned}
a_\kappa &= e^{2i\delta_\kappa} - 1 = (-1)^{l+1}\left(\frac{-\kappa + i\nu'}{\gamma + i\nu}\right)\frac{\Gamma(\gamma - i\nu)}{\Gamma(\gamma + i\nu)}e^{-i\pi\gamma} - 1 \\
&= (-\kappa + i\nu')\frac{\Gamma(\gamma - i\nu)}{\Gamma(\gamma + 1 + i\nu)}(-1)^{l+1}e^{-i\pi\gamma} - 1\,.
\end{aligned}
$$

Dropping terms of order $\alpha^2 Z^2$, we may approximate γ by $|\kappa|$ to obtain:

$$
\begin{aligned}
a_{-l-1} &= (l+1+i\nu')\,\frac{\Gamma(l+1-i\nu)}{\Gamma(l+2+i\nu)} - 1 \\
&= \left(\frac{\Gamma(l+1-i\nu)}{\Gamma(l+1+i\nu)} - 1\right) - i(\nu - \nu')\,\frac{\Gamma(l+1-i\nu)}{\Gamma(l+2+i\nu)} \\
a_l &= (l - i\nu')\,\frac{\Gamma(l-i\nu)}{\Gamma(l+1+i\nu)} - 1 \\
&= \left(\frac{\Gamma(l+1-i\nu)}{\Gamma(l+1+i\nu)} - 1\right) + i(\nu - \nu')\,\frac{\Gamma(l-i\nu)}{\Gamma(l+1+i\nu)}
\end{aligned}
$$

In the extreme nonrelativistic limit, $E \to m$ and $\nu \to \nu'$, leading to the limiting values $a_{-l-1} = a_l = a_l^{\rm NR}$ with,

$$a_l^{\rm NR} = \left(\frac{\Gamma(l+1-i\nu)}{\Gamma(l+1+i\nu)} - 1 \right) \ .$$

The limiting amplitudes are,

$$\begin{aligned} f^{\rm NR}(\theta) &= \frac{1}{2ip} \sum_{l=0}^{\infty} (2l+1) \left(\frac{\Gamma(l+1-i\nu)}{\Gamma(l+1+i\nu)} - 1 \right) P_l(\cos\theta) \\ g^{\rm NR}(\theta) &= 0 \ . \end{aligned}$$

We can evaluate the amplitudes in closed form without going to the extreme NR limit: First, note that:

$$\begin{aligned} (l+1)\frac{\Gamma(l+1-i\nu)}{\Gamma(l+2+i\nu)} - l\frac{\Gamma(l-i\nu)}{\Gamma(l+1+i\nu)} &= -i\nu(2l+1)\frac{\Gamma(l-i\nu)}{\Gamma(l+2+i\nu)} \\ \frac{\Gamma(l+1-i\nu)}{\Gamma(l+2+i\nu)} + \frac{\Gamma(l-i\nu)}{\Gamma(l+1+i\nu)} &= (2l+1)\frac{\Gamma(l-i\nu)}{\Gamma(l+2+i\nu)} \ . \end{aligned}$$

With the aid of these relations, we may write

$$\begin{aligned} f(\theta) &= f^{\rm NR}(\theta) - \frac{\nu(\nu-\nu')}{2ip} \sum_{l=0}^{\infty} (2l+1)\frac{\Gamma(l-i\nu)}{\Gamma(l+2+i\nu)} P_l(\cos\theta) \\ g(\theta) &= -\frac{(\nu-\nu')}{2p} \sum_{l=0}^{\infty} (2l+1)\frac{\Gamma(l-i\nu)}{\Gamma(l+2+i\nu)} P_l^1(\theta) \ . \end{aligned}$$

The sums above may be carried out in closed form. Introducing the two auxiliary functions $A(\theta)$ and $B(\theta)$,

$$\begin{aligned} A(\theta) &= \sum_{l=0}^{\infty} (2l+1) \left(\frac{\Gamma(l+1-i\nu)}{\Gamma(l+1+i\nu)} - 1 \right) P_l(\cos\theta) \\ B(\theta) &= \sum_{l=0}^{\infty} (2l+1)\frac{\Gamma(l-i\nu)}{\Gamma(l+2+i\nu)} P_l(\theta) \ , \end{aligned}$$

one finds

$$\begin{aligned} A(\theta) &= \frac{i\nu}{\sin^2(\theta/2)} \, \frac{\Gamma(1-i\nu)}{\Gamma(1+i\nu)} \, e^{i\nu \ln \sin^2(\theta/2)} \\ B(\theta) &= \frac{i}{\nu} \, \frac{\Gamma(1-i\nu)}{\Gamma(1+i\nu)} \, e^{i\nu \ln \sin^2(\theta/2)} \ . \end{aligned}$$

Using the fact that

$$P_l^1(\theta) = \sin\theta \, \frac{dP_l(\cos\theta)}{d\cos\theta}$$

and that

$$\sin\theta \, \frac{dB}{d\cos\theta} = \csc(\theta/2) \, \frac{\Gamma(1-i\nu)}{\Gamma(1+i\nu)} \, e^{i\nu \ln \sin^2(\theta/2)} \, ,$$

we obtain

$$\begin{aligned} f(\theta) &= \frac{\Gamma(1-i\nu)}{\Gamma(1+i\nu)} \, e^{i\nu \ln \sin^2(\theta/2)} \left[\frac{\nu}{2p} \csc^2(\theta/2) + \frac{\nu' - \nu}{2p} \right] \\ g(\theta) &= \frac{\Gamma(1-i\nu)}{\Gamma(1+i\nu)} \, e^{i\nu \ln \sin^2(\theta/2)} \left[\frac{\nu' - \nu}{2p} \cot(\theta/2) \right] \, . \end{aligned}$$

The differential scattering cross section (Mott cross section) is

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= |f|^2 + |g|^2 \\ &= \left(\frac{\nu^2}{4p^2} \right) \csc^4(\theta/2) \left\{ \left[1 + \left(\frac{m-E}{E} \right) \sin^2(\theta/2) \right]^2 \right. \\ &\qquad \left. + \left(\frac{m-E}{E} \right)^2 \sin^2(\theta/2) \cos^2(\theta/2) \right\} \\ &= \left(\frac{\alpha Z}{2pv} \right)^2 \, \frac{(1 - v^2 \sin^2(\theta/2))}{\sin^4(\theta/2)} \, . \end{aligned}$$

In the nonrelativistic limit, $v^2 \to 0$, the Mott cross section reduces to the Rutherford cross section. The leading correction, in powers of αZ, to the Mott cross section was found in W. R. Johnson, T. A. Weber, and C. J. Mullin, Phys. Rev. **121**, 933 (1961).