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Analysis of switched normal discrete-time systems

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Abstract

In this paper, we study stability and $\mathcal{L}_2$ gain properties for a class of switched systems which are composed of *normal* discrete-time subsystems. When all subsystems are Schur stable, we show that a common quadratic Lyapunov function exists for all subsystems and that the switched normal system is exponentially stable under arbitrary switching. For $\mathcal{L}_2$ gain analysis, we introduce an expanded matrix including each subsystem's coefficient matrices. Then, we show that if the expanded matrix is normal and Schur stable so that each subsystem is Schur stable and has unity $\mathcal{L}_2$ gain, then the switched normal system also has unity $\mathcal{L}_2$ gain under arbitrary switching. The key point is establishing a common quadratic Lyapunov function for all subsystems in the sense of unity $\mathcal{L}_2$ gain.
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1. Introduction

In the last two decades, there has been increasing interest in stability analysis and controller design for switched systems; see the survey papers [1–3], the recent books [4,5] and the references cited therein. The motivation for studying switched systems is from many aspects. It is known that many practical systems are inherently multimodal in the sense that several dynamical subsystems are required to describe their behavior which may depend on various

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environmental factors. Since these systems are essentially switched systems, powerful analysis or design results of switched systems are helpful for dealing with real systems. Another important observation is that switching among a set of controllers for a specified system can be regarded as a switched system, and that switching has been used in adaptive control to assure stability in situations where stability cannot be proved otherwise [6,7], or to improve transient response of adaptive control systems [8]. Also, the methods of intelligent control design are based on the idea of switching among different controllers [9]. Therefore, study on switched systems contributes greatly in switching controller and intelligent controller design.

When focusing on stability analysis of switched systems, there are three basic problems in stability and design of switched systems: (i) find conditions for stability under arbitrary switching; (ii) identify the limited but useful class of stabilizing switching laws; and (iii) construct a stabilizing switching law. There are many existing works on these problems in the case where the switched systems are composed of continuous-time subsystems. For Problem (i), Ref. [10] showed that when all subsystems are stable and pairwise commutative, the switched linear system is stable under arbitrary switching. Ref. [11] extended this result from the commutation condition to a Lie-algebraic condition. Ref. [12] showed that a class of symmetric switched systems are asymptotically stable under arbitrary switching since a common quadratic Lyapunov function, in the form of $V(x) = x^{\mathrm{T}}x$, exists for all the subsystems. Refs. [13, 14] considered Problem (ii) using piecewise Lyapunov functions, and Ref. [15] considered Problem (ii) for switched systems with pairwise commutation or Lie-algebraic properties. Ref. [16] considered Problem (iii) by dividing the state space associated with appropriate switching depending on state, and Ref. [17] considered quadratic stabilization, which belongs to Problem (iii), for switched systems composed of a pair of unstable linear subsystems by using a linear stable combination of unstable subsystems. Related to both Problems (ii) and (iii), Ref. [18] presented the convergence rate evaluation for simultaneously triangularizable switched systems, and Ref. [19] investigated the controllability and reachability of switched linear control systems. As regards the robustness stability/stabilization issue, Ref. [20] considered quadratic stabilizability of switched linear systems with polytopic uncertainties, and Ref. [21] dealt with robust quadratic stabilization for switched LTI systems by using piecewise quadratic Lyapunov functions so that the synthesis problem can be formulated as a matrix inequality feasibility problem. Refs. [22,12,23] extended the consideration to stability analysis problems for switched systems composed of discrete-time subsystems.

Motivated by the observation that all these papers deal with switched systems composed of only continuous-time subsystems or only discrete-time ones, the authors considered in [24] the new type of switched systems which are composed of both continuous-time and discrete-time dynamical subsystems, and gave some analysis and design results for several kinds of such switched systems, for example, the case where the commutation condition holds, and the case of switched symmetric systems. Recently, the authors extended the results for switched symmetric systems in [24] to switched *normal* systems in [25]. For such switched systems, it is shown that when all continuous-time subsystems are Hurwitz stable and all discrete-time subsystems are Schur stable, a common quadratic Lyapunov function exists for the subsystems and that the switched system is exponentially stable under arbitrary switching. Some discussions are also given for the case where unstable subsystems are involved.

In this paper, we focus our attention on switched systems which are composed of normal discrete-time subsystems. For such switched systems, we show that if all subsystems are Schur stable, then the switched system is exponentially stable under arbitrary switching. The main contribution of this paper is extending the consideration to $\mathcal{L}_2$ gain analysis for such switched systems. For this purpose, we introduce an expanded matrix including each subsystem's coefficient matrices. Then, we show that if the expanded matrix is normal and Schur stable so that each subsystem is Schur stable and has unity $\mathcal{L}_2$ gain, then there is a common quadratic Lyapunov function for all subsystems in the sense of $\mathcal{L}_2$ gain, and the switched normal system is asymptotically stable and also has unity $\mathcal{L}_2$ gain under arbitrary switching. As can be seen later, the normal assumption using the expanded matrix covers the case of switched symmetric systems that we dealt with in [23,12], and thus the result of $\mathcal{L}_2$ gain analysis here is a nontrivial extension of the existing works.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries about normal systems and state the Bounded Real Lemma for discrete-time LTI systems. In Section 3, we state and prove that if all subsystems are Schur stable and normal, then the switched normal system is exponentially stable under arbitrary switching. Two numerical examples are given to demonstrate the effectiveness and the applicability of the result. Section 4 is devoted to $\mathcal{L}_2$ gain analysis. We prove that if all subsystems are normal in the sense of unity $\mathcal{L}_2$ gain, then there is a common quadratic Lyapunov function for all subsystems in the sense of unity $\mathcal{L}_2$ gain, and thus the switched normal system also achieves unity $\mathcal{L}_2$ gain under arbitrary switching. Finally, Section 5 concludes the paper.

Notation: For a vector $x \in \mathbb{R}^n$, we use $|x|$ to denote its Euclidean norm $\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. For a symmetric matrix Q, we denote its maximum (minimum) eigenvalue as $\lambda_M(Q)$ ($\lambda_m(Q)$), use $Q > 0$ ($Q \geq 0$) when Q is positive definite (semi-positive definite), and $Q < 0$ ($Q \leq 0$) when Q is negative definite (semi-negative definite). For two symmetric matrices A, B, we use $A > B$ when $A - B$ is positive definite, and so on.

2. Preliminaries

We first give some definitions and lemmas concerning normal systems.

Definition 1. A discrete-time system

$$x(k+1) = Ax(k) \tag{1}$$

or the system matrix A is said to be *normal* if

$$A^{\mathrm{T}}A = AA^{\mathrm{T}}. \tag{2}$$

Definition 2. A real square matrix Q is said to be *orthogonal* if $Q^{\mathrm{T}}Q = I$.

The following lemma characterizes a normal system or matrix by orthogonally equalizing it to a block-diagonal matrix consisting of its eigenvalues (Theorem 4.10.69 in Ref. [26]).

Lemma 1. *Suppose that $A \in \Re^{n\times n}$ is normal, its real eigenvalues are $\lambda_1, \ldots, \lambda_r$, and its complex eigenvalues are $a_1 \pm b_1 i, \ldots, a_s \pm b_s i$, where a_i's and b_i's are real, $b_i \neq 0$, $r + 2s = n$. Then, there exists an orthogonal matrix Q such that*

$$Q^{\mathrm{T}}AQ = \mathrm{diag}\{\lambda_1, \ldots, \lambda_r, \Lambda_1, \ldots, \Lambda_s\}, \tag{3}$$

where

$$\Lambda_i = \begin{bmatrix} a_i & b_i \\ -b_i & a_i \end{bmatrix}, \quad i = 1, \ldots, s. \tag{4}$$

The following lemma plays a key role in this paper.

Lemma 2. *If the discrete-time system (1) is normal and Schur stable, then*

$$A^{\mathrm{T}}A - I < 0. \tag{5}$$

Proof. Using the fact that Q is orthogonal, we obtain from (3) that

$$\begin{aligned} Q^{\mathrm{T}}(A^{\mathrm{T}}A)Q &= (Q^{\mathrm{T}}A^{\mathrm{T}}Q)(Q^{\mathrm{T}}AQ) \\ &= \mathrm{diag}\{\lambda_1, \ldots, \lambda_r, \Lambda_1^{\mathrm{T}}, \ldots, \Lambda_s^{\mathrm{T}}\}\mathrm{diag}\{\lambda_1, \ldots, \lambda_r, \Lambda_1, \ldots, \Lambda_s\} \\ &= \mathrm{diag}\{\lambda_1^2, \ldots, \lambda_r^2, a_1^2 + b_1^2, a_1^2 + b_1^2, \ldots, a_s^2 + b_s^2, a_s^2 + b_s^2\}. \end{aligned} \tag{6}$$

Since A is Schur stable, we obtain $|\lambda_i| < 1$ $(1 \le i \le r)$ and $\sqrt{a_j^2 + b_j^2} < 1$ $(1 \le j \le s)$ and thus $Q^{\mathrm{T}}(A^{\mathrm{T}}A)Q < I$, which is equivalent to (5). □

The next lemma is on $\mathcal{L}_2$ gain analysis for discrete-time LTI systems, which we will use in Section 4.

Lemma 3 (*[27]*). *Consider the discrete-time LTI system*

$$\begin{aligned} x(k+1) &= Ax(k) + Bw(k) \\ z(k) &= Cx(k) + Dw(k), \end{aligned} \tag{7}$$

where $x(k) \in \Re^n$ is the system state, $w(k) \in \Re^m$ is the input, $z(k) \in \Re^p$ is the output, A, B, C, D are constant matrices of appropriate dimensions. The system (7) is Schur stable and has unity $\mathcal{L}_2$ gain (i.e., $\mathcal{L}_2$ gain less than 1) if and only if there exists $P > 0$ satisfying the LMI

$$\begin{bmatrix} A^{\mathrm{T}}PA - P + C^{\mathrm{T}}C & A^{\mathrm{T}}PB + C^{\mathrm{T}}D \\ B^{\mathrm{T}}PA + D^{\mathrm{T}}C & B^{\mathrm{T}}PB - I + D^{\mathrm{T}}D \end{bmatrix} < 0 \tag{8}$$

or equivalently

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} < \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}. \tag{9}$$

3. Stability analysis

In this section, we consider stability for the switched system which is composed of a set of discrete-time subsystems described by

$$x(k+1) = A_i x(k), \quad i = 1, \ldots, N, \tag{10}$$

where $x(k) \in \Re^n$ is the subsystem state, A_i $(i = 1, \ldots, N)$ are constant matrices of appropriate dimensions denoting the subsystems, and $N \ge 1$ denotes the number of subsystems.

There are two important basic assumptions throughout this paper. First, there is no state jumping occurring at switching instants. The reason is that in the case where state jumping occurs with an increase of the state's norm, arbitrary switching is not possible. If we know that the state jumping only leads to decrease of the state's norm, then all the discussion in this paper is the same. Second, we assume that only one subsystem is being activated at any time instant, and thus we do not have to consider the synchronization issue even if the subsystems have different sampling periods. This is reasonable in real applications. For example, in a switching control problem involving an open-loop system and several feedback controllers, we usually generate the control input by choosing only one feedback controller and do not let all the other feedback controllers work for all time. The second assumption is necessary for the first one, since state jumping may occur when the subsystems have different sampling periods and synchronization is not done exactly at the switching instants. However, we note that for general design problems of switched discrete-time systems, synchronization between sampling time and switching time is an important issue that has to be discussed carefully.

Since arbitrary switching includes the case of dwelling on certain subsystem for all time, we make the following necessary assumption.

Assumption 1. All A_i's are Schur stable.

It is known that Assumption 1 is not enough to guarantee stability under arbitrary switching. That is, a switched system composed of stable subsystems could be unstable if the switching is not done appropriately [1]. In [24], we considered two conditions, namely, the "commutation condition" and "symmetricity condition", under which arbitrary switching is possible. Here, we extend the latter condition by making the following assumption.

Assumption 2. All the subsystems in (10) are *normal*, i.e.,

$$A_i^{\mathrm{T}} A_i = A_i A_i^{\mathrm{T}}. \tag{11}$$

Remark 1. For switched symmetric systems, it is assumed in [23,24] that $A_i^{\mathrm{T}} = A_i$. Obviously, Assumption 2 covers such symmetric systems. Furthermore, it covers the cases of $A_i^{\mathrm{T}} A_i = I$, $A_i^{\mathrm{T}} = -A_i$ and some other cases.

We now state and prove the first result.

Theorem 1. *Under Assumptions 1 and 2, the switched system composed of (10) is exponentially stable under arbitrary switching.*

Proof. Since all the subsystems are normal, according to Lemmas 1 and 2, we obtain

$$A_i^{\mathrm{T}} A_i - I < 0, \quad i = 1, \ldots, N. \tag{12}$$

This implies that $P = I$ is a common solution to the Lyapunov matrix inequalities

$$A_i^{\mathrm{T}} P A_i - P < 0, \quad i = 1, \ldots, N, \tag{13}$$

and thus $V(x) = x^{\mathrm{T}} x$ is a common quadratic Lyapunov function for all the subsystems.

To show the exponential stability of the system, we find a positive scalar $\alpha < 1$ such that

$$A_i^{\mathrm{T}} A_i - \alpha^2 I < 0 \tag{14}$$

holds for all i's. Such an α always exists; for example, we can choose that $\alpha = \max_{1 \le i \le N} \{\lambda_M(A_i^{\mathrm{T}} A_i)\}$. Then, in any time interval, we obtain $V(x(k+1)) < \alpha^2 V(x(k))$. Since all the subsystems share the Lyapunov function candidate, we obtain for any $k \ge 0$ that

$$V(x(k)) \le \alpha^{2k} V(x(0)) = \mathrm{e}^{-(2\ln(\alpha^{-1}))k} V(x(0)) \tag{15}$$

and thus

$$|x(k)| \le \mathrm{e}^{-(\ln(\alpha^{-1}))k} |x(0)|. \tag{16}$$

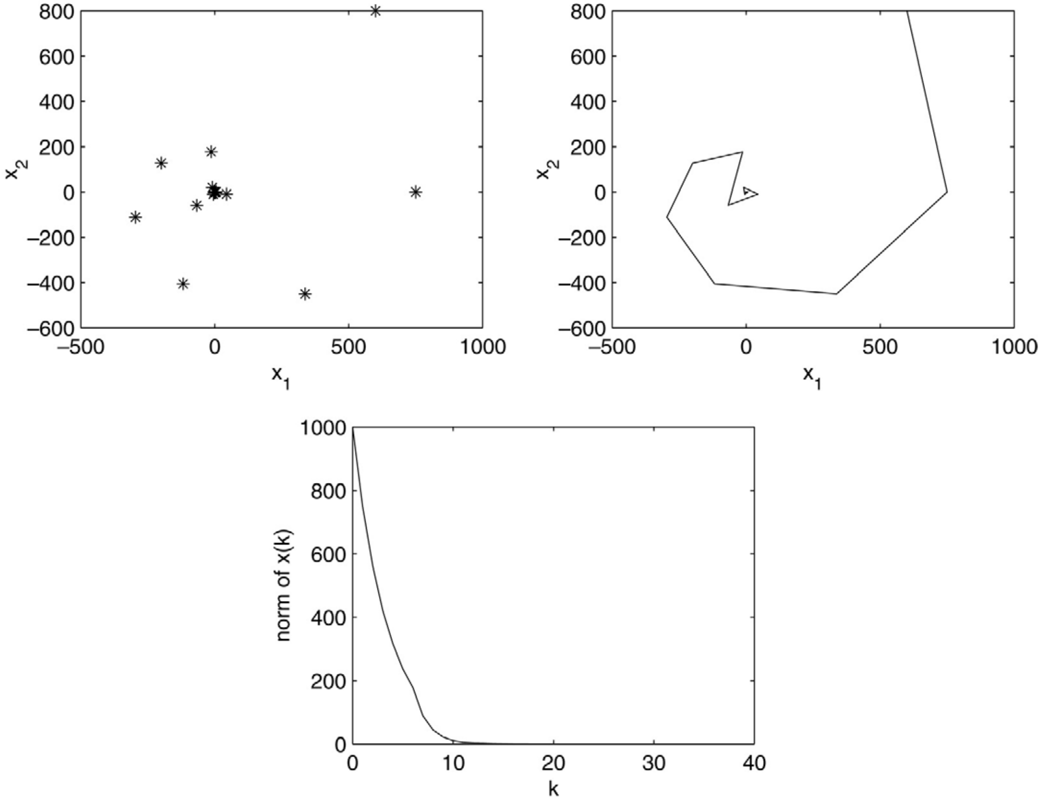

Fig. 1. The system trajectory and the state's norm in Example 1.

Noting that we did not add any condition on the switching signal, the switched system is exponentially stable under arbitrary switching. □

Remark 2. It has been shown in the proof of Theorem 1 that when all subsystems are normal and Schur stable, $V(x) = x^T x$ is a common quadratic Lyapunov function for them.

Example 1. Consider the switched system composed of two subsystems given by

$$A_1 = \begin{bmatrix} 0.45 & 0.6 \\ -0.6 & 0.45 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.3 & -0.4 \\ 0.4 & -0.3 \end{bmatrix}. \tag{17}$$

It is easy to confirm that both A_1 and A_2 are normal and Schur stable. Fig. 1 shows the convergence of the system trajectory where A_1 and A_2 are activated alternatively with a randomly determined steps of (6, 5, 9, 3, 4, 7). The initial state is $[600\ 800]^T$, and the mark "*" in the upper left part of Fig. 1 describes the state change, while the upper right part of Fig. 1 connects all the sampling points into a continuous trajectory. The lower part of Fig. 1 shows that the norm of the system state converges to zero very quickly.

At the end of this section, we note that Theorem 1 is useful in many switching control problems. Suppose that we have on hand an open-loop feedback system

$$x(k+1) = Ax(k) + Bu(k) \tag{18}$$

where $x(k)$ is the state, $u(k)$ is the input, A, B are constant matrices of appropriate dimension. We also suppose that we can design a set of state feedback controllers $u(k) = F_i x(k)$ ($i = 1, \ldots, N_m$), such that each $A + BF_i$ is normal and Schur stable. This is possible in many cases. For example, when

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{19}$$

it is easy to know that any feedback gain $F = \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}$, with f_i's satisfying $f_1 + f_4 = 3$, $f_3 - f_2 = 5$, $(f_1 + 1)^2 + (f_2 + 3)^2 < 1$, will make $A + BF$ normal and Schur stable. In fact, there are many F's satisfying this condition.

If we can (or have to) choose one from the set of controllers at every time instant, the whole system is a switched system that is composed of Schur stable subsystems. Then, according to Theorem 1, we see that the system is exponentially stable no matter how we choose the controllers. This observation is very important in real applications when we want more flexibility to take other specification into consideration.

Obviously, the above discussion is also applicable to the case of output feedback switching control problems. Furthermore, a more interesting problem may be feedback control systems which are composed of a continuous-time plant and discrete-time controllers.

Example 2. For the system (18) with (19), we set

$$F_1 = \begin{bmatrix} -0.5 & -2.5 \\ 2.5 & 3.5 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -0.8 & -3.6 \\ 1.4 & 3.8 \end{bmatrix} \tag{20}$$

to obtain two closed-loop system matrices

$$A_1 = \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.2 & -0.6 \\ 0.6 & 0.2 \end{bmatrix} \tag{21}$$

which are normal and Schur stable.

Now, we set the initial state as $x_0 = [600\ 800]^{\mathrm{T}}$ and the two controllers are activated alternatively with three steps and two steps, respectively. Fig. 2 shows the convergence of the system trajectory under such a switching method.

In this section, we have focused our attention on stability analysis of the switched normal systems (10) under arbitrary switching, i.e., in the framework of the basic problem (i) which we mentioned in the introduction. As regards Problem (ii) and (iii) for switching method design, we have given some discussion in [25], which may be referred to for details.

4. $\mathcal{L}_2$ gain analysis

In this section, we consider the $\mathcal{L}_2$ gain property for the switched system which is composed of discrete-time subsystems described by

$$\begin{aligned} x(k+1) &= A_i x(k) + B_i w(k) \\ z(k) &= C_i x(k) + D_i w(k), \quad i = 1, \ldots, N, \end{aligned} \tag{22}$$

where $x(k) \in \Re^n$ is the subsystem state, $w(k) \in \Re^m$ is the input, $z(k) \in \Re^p$ is the output. A_i, B_i, C_i, D_i ($i = 1, \ldots, N$) are constant matrices of appropriate dimensions denoting the subsystems,

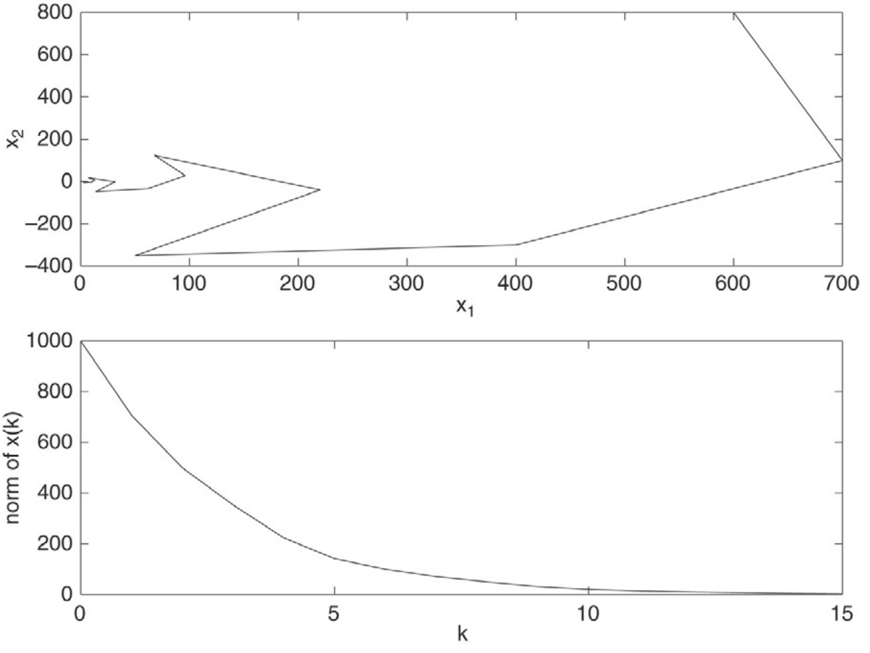

Fig. 2. The system trajectory and the state's norm in Example 2.

and $N > 1$ is the number of subsystems. Although the discussion can be easily extended to the case of $x[0] \neq 0$, we assume for brevity that $x[0] = 0$ in this section.

Furthermore, we assume $m = p$, and define the square matrix

$$G_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \tag{23}$$

to include each subsystem's coefficient matrices. Throughout this section, we make the following assumption.

Assumption 3. All G_i's are Schur stable and are *normal*, i.e.,

$$G_i^{\mathrm{T}} G_i = G_i G_i^{\mathrm{T}}. \tag{24}$$

Remark 3. The condition (24) is computed as

$$\begin{bmatrix} A_i^{\mathrm{T}} A_i + C_i^{\mathrm{T}} C_i & A_i^{\mathrm{T}} B_i + C_i^{\mathrm{T}} D_i \\ B_i^{\mathrm{T}} A_i + D_i^{\mathrm{T}} C_i & B_i^{\mathrm{T}} B_i + D_i^{\mathrm{T}} D_i \end{bmatrix} = \begin{bmatrix} A_i A_i^{\mathrm{T}} + B_i B_i^{\mathrm{T}} & A_i C_i^{\mathrm{T}} + B_i D_i^{\mathrm{T}} \\ C_i A_i^{\mathrm{T}} + D_i B_i^{\mathrm{T}} & C_i C_i^{\mathrm{T}} + D_i D_i^{\mathrm{T}} \end{bmatrix} \tag{25}$$

which requires

$$\begin{aligned} A_i^{\mathrm{T}} A_i + C_i^{\mathrm{T}} C_i &= A_i A_i^{\mathrm{T}} + B_i B_i^{\mathrm{T}} \\ A_i^{\mathrm{T}} B_i + C_i^{\mathrm{T}} D_i &= A_i C_i^{\mathrm{T}} + B_i D_i^{\mathrm{T}} \\ B_i^{\mathrm{T}} B_i + D_i^{\mathrm{T}} D_i &= C_i C_i^{\mathrm{T}} + D_i D_i^{\mathrm{T}}. \end{aligned} \tag{26}$$

This covers the following symmetricity condition that we have assumed in the existing works [23,12]:

$$A_i = A_i^{\mathrm{T}}, \quad B_i = C_i^{\mathrm{T}}, \quad D_i = D_i^{\mathrm{T}}. \tag{27}$$

In other words, the symmetric switched systems satisfying (27), which we have considered in [23, 12], all satisfy the normal condition (24).

Furthermore, the condition (24) or (25) also includes the case of skew-symmetric systems described by

$$A_i = -A_i^{\mathrm{T}}, \quad B_i = -C_i^{\mathrm{T}}, \quad D_i = -D_i^{\mathrm{T}}. \tag{28}$$

Since G_i is normal and Schur stable, according to Lemma 2, we obtain

$$G_i^{\mathrm{T}} G_i - I < 0 \tag{29}$$

which is equivalent to

$$\begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} < \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{30}$$

Comparing the above inequality with the matrix inequality (9) in Lemma 3, we obtain that under Assumption 3, the ith subsystem is Schur stable and has unity $\mathcal{L}_2$ gain with $P = I$ satisfying the LMI (9). In this case, we say that the subsystems in (22) are *normal in the sense of unity* $\mathcal{L}_2$ *gain*. Since this fact is true for all G_i's, we see that all subsystems have a common quadratic Lyapunov function $V(x) = x^{\mathrm{T}} x$ in the sense of unity $\mathcal{L}_2$ gain.

We now compute the difference of the Lyapunov function $V(x) = x^{\mathrm{T}} x$ along the trajectory of any subsystem to obtain

$$\begin{aligned} & V(x(k+1)) - V(x(k)) \\ &\quad = x^{\mathrm{T}}(k+1)x(k+1) - x^{\mathrm{T}}(k)x(k) \\ &\quad = (A_i x(k) + B_i w(k))^{\mathrm{T}} (A_i x(k) + B_i w(k)) - x^{\mathrm{T}}(k)x(k) \\ &\quad = \begin{bmatrix} x^{\mathrm{T}}(k) & w^{\mathrm{T}}(k) \end{bmatrix} \begin{bmatrix} A_i^{\mathrm{T}} A_i - I & A_i^{\mathrm{T}} B_i \\ B_i^{\mathrm{T}} A_i & B_i^{\mathrm{T}} B_i \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \\ &\quad \le - \begin{bmatrix} x^{\mathrm{T}}(k) & w^{\mathrm{T}}(k) \end{bmatrix} \begin{bmatrix} C_i^{\mathrm{T}} C_i & C_i^{\mathrm{T}} D_i \\ D_i^{\mathrm{T}} C_i & D_i^{\mathrm{T}} D_i - I \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \\ &\quad = - \left(z^{\mathrm{T}}(k) z(k) - w^{\mathrm{T}}(k) w(k) \right), \end{aligned} \tag{31}$$

where (30) was used to obtain the inequality, and the inequality holds strictly when either $x(k)$ or $w(k)$ is not zero.

For an arbitrary piecewise constant switching signal and any given integer $k > 0$, we let $k_1, \ldots, k_r$ $(r \ge 1)$ denote the switching points over the interval $[0, k)$. Then, using the difference inequality (31), we obtain

$$
\begin{aligned}
V(x(k)) - V(x(k_r)) &\le -\sum_{j=k_r}^{k-1} \Gamma(j)\\
V(x(k_r)) - V(x(k_{r-1})) &\le -\sum_{j=k_{r-1}}^{k_r-1} \Gamma(j)\\
&\cdots\cdots\cdots\cdots\\
V(x(k_1)) - V(x(0)) &\le -\sum_{j=0}^{k_1-1} \Gamma(j),
\end{aligned}
\tag{32}
$$

where $\Gamma(j) \triangleq z^{\mathrm{T}}(j)z(j) - w^{\mathrm{T}}(j)w(j)$. Since the case of $x(j) \equiv 0$, $w(j) \equiv 0$, $0 \le j \le k$, is a trivial one and is thus excluded in our $\mathcal{L}_2$ gain analysis, there is at least one of the inequalities in (32) that should hold strictly (i.e., without "="). We add all the inequalities to get to

$$
V(x(k)) - V(x(0)) < -\sum_{j=0}^{k-1} \Gamma(j). \tag{33}
$$

In this inequality, we use the assumption that $x(0) = 0$ and the fact that $V(x(k)) \ge 0$ to obtain

$$
\sum_{j=0}^{k-1} z^{\mathrm{T}}(j)z(j) < \sum_{j=0}^{k-1} w^{\mathrm{T}}(j)w(j), \tag{34}
$$

which implies that unity $\mathcal{L}_2$ gain is achieved. Since the above inequality holds for any $k > 0$ including the case of $k \to \infty$, and there is not any restriction added on the switching signal (the switching signal can be arbitrary), we say that the switched system achieves *unity $\mathcal{L}_2$ gain under arbitrary switching*.

We summarize the above discussion in the following theorem.

Theorem 2. *If all subsystems in (22) are normal in the sense of unity $\mathcal{L}_2$ gain (satisfying Assumption 3), then there is a common quadratic Lyapunov function $V(x) = x^{\mathrm{T}}x$ for all subsystems in the sense of unity $\mathcal{L}_2$ gain, and thus the switched normal system (22) achieves unity $\mathcal{L}_2$ gain under arbitrary switching.* □

Remark 4. For brevity, we consider unity $\mathcal{L}_2$ gain in this section. In the case of $\mathcal{L}_2$ gain γ, since the LMI (9) in Lemma 3 takes the form of

$$
\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} < \begin{bmatrix} P & 0 \\ 0 & \gamma^2 I \end{bmatrix}, \tag{35}
$$

what we have to do is to replace G_i of (23) with

$$
G_{i\gamma} = \begin{bmatrix} A_i & \dfrac{1}{\sqrt{\gamma}} B_i \\ \dfrac{1}{\sqrt{\gamma}} C_i & \dfrac{1}{\gamma} D_i \end{bmatrix}. \tag{36}
$$

In the case where each subsystem has a different $\mathcal{L}_2$ gain γ_i, we define $\gamma = \max_i \gamma_i$ and proceed in the same way.

Finally, we relax Assumption 3 slightly by making the following assumption instead.

Assumption 3′. All G_i's are neutrally Schur stable ($G_i^{\mathrm{T}} G_i \leq I$) and are normal, satisfying (24).

Under Assumption 3′, the matrix $P = I$ satisfies the nonstrict LMI

$$\begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \leq \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \tag{37}$$

for all $i = 1, \ldots, N$.

Using the same discussion as in Theorem 2, we obtain that under arbitrary switching, the inequality

$$\sum_{j=0}^{k-1} z^{\mathrm{T}}(j) z(j) \leq \sum_{j=0}^{k-1} w^{\mathrm{T}}(j) w(j) \tag{38}$$

holds for any $k > 0$, which implies that nonstrict unity $\mathcal{L}_2$ gain is achieved.

Corollary 1. *If all subsystems in (22) are normal in the sense of nonstrict unity $\mathcal{L}_2$ gain (satisfying Assumption 3′), then there is a common quadratic Lyapunov function $V(x) = x^{\mathrm{T}} x$ for all subsystems in the sense of $\mathcal{L}_2$ gain, and thus the switched normal system (22) also achieves nonstrict unity $\mathcal{L}_2$ gain under arbitrary switching.* □

5. Conclusion

In this paper, we have studied stability and $\mathcal{L}_2$ gain properties for a class of switched systems which are composed of normal discrete-time subsystems. When all subsystems are Schur stable, we have shown that $V(x) = x^{\mathrm{T}} x$ is a common quadratic Lyapunov function for the subsystems and that the switched normal system is exponentially stable under arbitrary switching. As regards $\mathcal{L}_2$ gain analysis, we have introduced an expanded matrix including each subsystem's coefficient matrices, and have shown that if the expanded matrix is normal and Schur stable so that each subsystem is Schur stable and has unity $\mathcal{L}_2$ gain, then the switched normal system also has unity $\mathcal{L}_2$ gain under arbitrary switching. The key point is establishing a common quadratic Lyapunov function for all subsystems in the sense of unity $\mathcal{L}_2$ gain.

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