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New Results on Controllability of Multi-Agent Systems

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***Abstract*— In this paper, the controllability problem was studied for multi-agent systems under leader-follower structure. The contribution includes: (i) the controllability analysis for a kind of directed interconnection topology, i.e, the so-called non-fully symmetric (NFS) directed interconnection graph; (ii) the presentation of some connections between controllability and consensus; (iii) the changing of a class of uncontrollable undirected interconnection graph into a controllable one by assigning weights between agents. Also, the selection of leaders is discussed. Several examples are given to illustrate the ideas and the corresponding results.**

I. Introduction

Recently, the distributed coordination and control of multi-agent networked systems has received considerable attention. The coordination and cooperation between agents need communications among them. The characterization of properties of such systems then relies closely on the topology structure of the graph associated with the network. This motives the study of controllability related interconnection topologies for multi-agent systems in the paper.

The controllability problem of networked multi-agent systems was proposed by Tanner for the investigation of formation control under leader-follower framework [1]. Although the study of controllability is the core of classical control, few results were reported along this line for multi-agent systems except e.g., [2], [3], [4], [5]. Consensus is another well recognized problem in the literature. Many results on consensus have been obtained, e.g, [6], [7], [8]. In this paper, some connections between consensus and controllability are given as well as the controllability and uncontrollability characterizations for a kind of directed/undirected interconnection graphs. Several examples are also constructed to illustrate the problem and idea from different viewpoints.

II. Preliminaries

A directed graph (undirected graph) $\mathcal{G}$ consists of a vertex set $\mathcal{V}$ and an arc (edge) set ε, where an edge is an ordered (unordered) pair of distinct vertices in $\mathcal{G}$. If vertices $x, y \in \mathcal{V}$ and arc $(x, y) \in \varepsilon$, then x and y are said to be neighbors. Denote the neighbors of vertex v_i by $N_i = \{v_j : (v_j, v_i) \in \varepsilon\}$. A directed graph $\mathcal{G}$ is strongly connected if there exists a path between any two distinct vertices, and is weakly connected if any two vertices can be jointed by a weak path. We assume that there are no self-loops, and there are no multiple arcs between any pair of distinct vertices. The Laplacian matrix of a weighted graph $\mathcal{G}$ is defined as $L(a_{ij}) = L(\mathcal{G}) = \Delta - A$, where $\Delta = \text{diag}\{d_1, \cdots, d_n\}$, d_i is the degree of node v_i. A directed (undirected) tree is a directed (undirected) graph, where every vertex, except the root, has exactly one parent. A spanning tree of a directed (undirected) graph is a tree formed by arcs (edges) which connected all vertices of the graph.

Consider $N + m$ agents, in which the agents indexed by $N + i$ $(i = 1, \cdots, m)$ are assigned as leaders and the others indexed by $1, \cdots, N$ are referred to as followers. A continuous-system model of $N + m$ agents is described by:

$$\begin{cases} \dot{x}_i = u_i, & i = 1, \cdots N \\ \dot{x}_{N+j} = u_{N+j} & j = 1, \cdots, m \end{cases} \tag{1}$$

where $x_i \in R^n$ is the state of agent i, and $u_i \in R^n$ is the input, $i = 1, \cdots N + m$. Under the following protocol,

$$u_i = -\sum_{j \in N_i} a_{ij}(x_i - x_j), \quad i = 1, \cdots, N + m \tag{2}$$

the multi-agent system (1) reads $\dot{x} = -Lx$, where L is written as

$$l_{ij} = \begin{cases} -a_{ij}, & \text{if } i \neq j \text{ and } j \in N_i, \\ \sum\limits_{j \in N_i} a_{ij}, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases},$$

$A = [a_{ij}]$ is the weighted adjacency matrix, $x = [x_1, \cdots, x_{N+m}]^T$ is the stack vector of all the agents.

Take $x_{N+1}, \cdots, x_{N+m}$ to play leader's role, and assume that the followers still obey (2), but the leaders are indifferent, and are free to pick u_{N+j} arbitrarily, where $j = 1, \cdots, m$. Now, let us rewrite the agents as

$$\begin{cases} y_i \overset{\Delta}{=} x_i & i = 1, \cdots, N \\ z_j \overset{\Delta}{=} x_{N+j} & j = 1, \cdots, m \end{cases}$$

With y and z being the stack vectors of all followers y_i and leaders z_i, and u being the stack vector of all u_{N+j}, where $j = 1, \cdots, m$, we can rewrite system in the form:

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = -\begin{bmatrix} L_f & l_{fl} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}$$

where L_f is the matrix obtained from L after deleting the last m rows and m columns, and l_{fl} is the $N \times m$ submatrix consisting the first N elements of the deleted columns. Then the dynamics of the followers that correspond to the y component of the equation can be extracted as

$$\dot{y} = -L_f y - l_{fl} z \tag{3}$$

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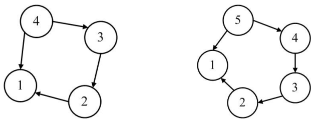
with the control inputs being the leaders' states.

Definition 1: The multi-agent system (1) is said to be controllable under leaders x_{N+j} and fixed topology if system (3) is controllable.

III. A DIRECTED TOPOLOGY STRUCTURE

We present a heuristic example at first.



(a) (b)

Fig. 1. (a) A directed graph with agent 4 playing the leader role; (b) A directed graph with agent 5 playing the leader role.

Example 1: The two interconnection graphs (a) and (b) in Fig.1 have the following common features: (i) With respect to each of them, there are only two followers N and 1 which can receive the information from the leader. (ii) The information flow between followers is in the same fashion. That is, $N \to N-1 \to \cdots \to 1$, where $N+1$ is the leader.

It can be verified that the multi-agent system associated with the interconnection graph (a) is controllable, while the one associated with (b) not.

A general directed interconnection graph can be defined in the same way as those indicated in Fig.1, which shares the aforementioned common features. We call this kind of graph the *non-fully symmetric* (NFS) directed interconnection graph. Below we consider the single leader case. It is assumed without loss of generality that the last agent, i.e. agent $N+1$ takes the leader role. In this case, the Laplacian matrix L is $(N+1)\times(N+1)$. It can be verified straightforward that the multi-agent system associated with NFS directed interconnection graph is uncontrollable when $N = 1, 2$. So the following conclusion deals with the case of $N \geq 3$.

Theorem 1: The multi-agent system (1) with NFS directed interconnection graph is controllable if N is an odd integer, and uncontrollable if N is an even integer, where $N \geq 3$.

Proof: The controllability matrix of system (3) is

$$C = \left[-l_{fl}, L_f l_{fl}, (-1)^3 L_f^2 l_{fl}, \cdots, (-1)^N L_f^{N-1} l_{fl}\right].$$

which will be calculated in the sequel. Since the directed interconnection graph is non-fully symmetric, one has

$$L_f = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \ddots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{N\times N}, \; l_{fl} = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}_{N\times 1}$$

Computations show that the entries of C have the following properties:

(i) $|C_{1,j}| = |C_{2,j}| + \cdots + |C_{N,j}|, \; j = 1, \cdots, N-1$, where $C_{i,j}$ is the entry of C located at the i-th row and j-th column, and $C_{N-1,j} \leq 0, C_{N-2,j} \geq 0, \cdots,$

$$C_{2,j} \begin{cases} \geq 0, & \text{if } N \text{ is an odd integer,} \\ \leq 0, & \text{if } N \text{ is an even integer.} \end{cases}$$

with $j = 1, \cdots, N$.

(ii) $C_{i,j} = C_{i,j-1} - C_{i+1,j-1}, \; i = 2, \cdots, N-1; j = 2, \cdots, N$, and

$$\begin{cases} C_{i,j} \neq 0, & \text{if } i \leq j; \; i,j = 1, \cdots N \\ C_{i,j} = 0, & \text{if } i > j; \; i = 2, \cdots, N; j = 1, \cdots N \end{cases}$$

(iii)

$$C_{1,N} = \begin{cases} -2^{N-1} + 1, & \text{if } N \text{ is an even integer,} \\ -2^{N-1} - 1, & \text{if } N \text{ is an odd integer.} \end{cases}$$

The remainder proof will be carried on through induction. Firstly, by Example 1, the result holds for $N = 3, 4$. Then, if n is an odd integer and the result holds for $N = n$ and $N = n+1$, we will show that it is also true for $N = n+2$ and $N = n+3$.

When $N = n+1$, with n being an odd integer, calculations show that the controllable matrix is

$$C = \begin{bmatrix} -1 & -2 & \cdots & \cdots & -2^{n-1} & \alpha \\ 0 & 0 & \cdots & \cdots & -1 & b_1 \\ 0 & 0 & \cdots & \iddots & a_1 & b_2 \\ \vdots & \vdots & \iddots & \cdots & \vdots & \vdots \\ 0 & 1 & \cdots & \cdots & a_{n-2} & b_{n-1} \\ -1 & -1 & \cdots & \cdots & -1 & -1 \end{bmatrix}$$

where a_i, b_i are entries which may be nonzero. It follows from Property (iii) that $\alpha = -2^n + 1$. By elementary row transformations, the controllable matrix C is equivalent to an anti-diagonal matrix, i.e.

$$C \sim \begin{bmatrix} & & & & x \\ & & & -1 & \\ & & \iddots & & \\ & 1 & & & \\ -1 & & & & \end{bmatrix} = K, \tag{4}$$

where '$\sim$' stands for the equivalence of two matrices in the sense of elementary transformation. By Property (i), one has

$$\begin{aligned} x =& \alpha - b_1 + b_2 - \cdots - b_{n-2} + b_{n-1} + 1 \\ =& -2^n - b_1 + b_2 - \cdots - b_{n-2} + b_{n-1} + 2. \end{aligned}$$

Sine $N = n+1$ is an even integer, it follows from the induction assumption that the multi-agent system is uncontrollable. This, together with (4), gives rise to $rank(C) = rank(K) = N-1$. As a consequence, one has $x = 0$ since the anti-diagonal elements of K are all nonzero except x.

When $N = n+2$, with n being an odd integer, one has

$$C = \begin{bmatrix} -1 & -2 & \cdots & \cdots & -2^n & \beta \\ 0 & 0 & \cdots & \cdots & 1 & c_1 \\ 0 & 0 & \cdots & \iddots & b_1 & c_2 \\ \vdots & \vdots & \iddots & \cdots & \vdots & \vdots \\ 0 & 1 & \cdots & \cdots & b_{n-1} & c_n \\ -1 & -1 & \cdots & \cdots & -1 & -1 \end{bmatrix},$$

where $\beta = -2^{n+1} - 1$. By the aforementioned Property (ii) of the entries of C, one has $c_1 = 1 - b_1, c_2 = b_1 - b_2, \cdots, c_{n-1} = b_{n-2} - b_{n-1}, c_n = b_{n-1} + 1$. By elementary row transformations, the controllable matrix C is equivalent to the following anti-diagonal matrix,

$$C \sim \begin{bmatrix} & & & & y \\ & & & 1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & 1 & & & \\ -1 & & & & \end{bmatrix}, \qquad (5)$$

where $y = -2^{n+1} - 1 + c_1 - c_2 + \cdots + c_n - (-1)$. It follows from $x = 0$ that

$$\begin{aligned} y =& -2^{n+1} - 1 + 1 - b_1 - b_1 + b_2 + b_2 - b_3 - \cdots + b_{n-1} \\ &+ 1 + 1 \\ =& -2^{n+1} + 2 - 2b_1 + 2b_2 - 2b_3 + \cdots + 2b_{n-1} \\ =& 2x - 2 = -2. \end{aligned}$$

Combining this with (5) yields that the system is controllable.

When $N = n + 3$, with n being an odd integer, one has

$$C = \begin{bmatrix} -1 & -2 & \cdots & \cdots & -2^{n+1} & \gamma \\ 0 & 0 & \cdots & \cdots & -1 & d_1 \\ 0 & 0 & \cdots & \cdot^{\cdot^{\cdot}} & c_1 & d_2 \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \cdots & \vdots & \vdots \\ 0 & 1 & \cdots & \cdots & c_n & d_{n+1} \\ -1 & -1 & \cdots & \cdots & -1 & -1 \end{bmatrix},$$

where $\gamma = -2^{n+2} + 1$. It can be verified that

$$C \sim \begin{bmatrix} & & & & z \\ & & & -1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & 1 & & & \\ -1 & & & & \end{bmatrix}, \qquad (6)$$

where $z = -2^{n+2} + 1 - d_1 + d_2 - \cdots - d_n + d_{n+1} - (-1)$. Because $d_1 = -1 - c_1 = -1 - (1 - b_1), d_2 = c_1 - c_2 = 1 - b_1 - (b_1 - b_2), \cdots, d_n = c_{n-1} - c_n = b_{n-2} - b_{n-1} - (b_{n-1} + 1), d_{n+1} = c_n + 1 = b_{n-1} + 1 + 1$, it follows from $x = 0$ again that

$$\begin{aligned} z =& -2^{n+2} + 1 + 1 + 1 - b_1 + 1 - b_1 - (b_1 - b_2) - \cdots - \\ & [b_{n-2} - b_{n-1} - (b_{n-1} + 1)] + b_{n-1} + 1 + 1 + 1 \\ =& -2^{n+2} + 8 - 4b_1 + 4b_2 - \cdots - 4b_{n-2} + 4b_{n-1} \\ =& 4x = 0. \end{aligned}$$

By (6), the system is uncontrollable. ■

IV. Connections between Controllability and Consensus

We study the connection between the controllability and consensus problem in this section.

Definition 2: [6] Let $x = [x_1^T, x_2^T, \cdots, x_{N+m}^T]$. Then the whole system can be generally represented by $\dot{x}(t) = u(t)$, where $u(t)$ is called the protocol. If for any initial state $x(t)$ converges to some equilibrium point x^* (dependent on the initial state) such that $x_i^* = x_j^*$ for all $i, j \in I_{N+m}$ ($I_{N+m} = \{1, 2, \cdots, N + m\}$) as $t \to \infty$, then we say that this system solves a consensus problem (or has consensus property).

The common value of x_i^* is called the group decision value. Given the initial states of agents convergent to a common value, we say this system will reach consensus asymptotically.

Lemma 1: [7] System (1) solves a consensus problem if and only if the associated interaction graph $\mathcal{G}$ has a spanning tree.

Lemma 2: (Corollary 2.4.1 in [10]) Every connected graph contains a spanning tree.

Lemma 3: [7] Given a matrix $S = [a_{ij}] \in R^{n \times n}$, where $a_{ii} \geq 0, a_{ij} \leq 0, \forall i \neq j$ and $\sum_{j=1}^{n} a_{ij} = 0$ for each j, then S has at least one zero eigenvalue and all of the nonzero eigenvalues are in the open right half plane. Furthermore, S has exactly one zero eigenvalue if and only if the graph with S has a spanning tree.

Definition 3: (leader-follower connected topology [5]) An interconnection graph $\mathcal{G}$ is said to be leader-follower connected if for each connected component $\mathcal{G}_{c_i}$ in the follower subgraph $\mathcal{G}_f$, there exists a leader in the leader subgraph $\mathcal{G}_l$, so that there is an edge between this leader and a node in $\mathcal{G}_{c_i}$, $i = 1, \cdots, \gamma$.

Lemma 4: (Lemma 3 in [5]) If multi-agent system (3) with fixed topology is controllable, the interconnection graph $\mathcal{G}$ is leader-follower connected.

Remark 1: It should be noted that although Lemma 4 was proved for undirected interconnection graph in [5], it still holds for any directed one, which can be verified by repeating the same lines of the proof as those in [5]. With respect to directed graph, the connectedness between leaders and followers subgraphs means weak connectedness.

Lemma 5: (Theorem 9.1.1 in [9]) Suppose S is a real symmetric $n \times n$ matrix and T is a sub-matrix of S with order $m \times m$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ be the respective eigenvalues of S and T. Then for $i \in 1, \cdots, m$,

$$\lambda_{n-m+i} \leq \mu_i \leq \lambda_i \qquad (7)$$

Theorem 2: For a multi-agent system, the following properties hold:

(i) If a multi-agent system with single leader is controllable, it solves a consensus problem; while this may not be the case if leaders are multiple.

(ii) If a multi-agent system with single leader and undirected interconnection graph does not solve a consensus problem, the system matrix L_f and the Laplacian matrix L share a common eigenvalue zero.

(iii) An uncontrollable multi-agent system can be turned into a controllable one by selecting appropriate weights for communication links; while the consensus property is invariant under different weights.

Proof: Part (i) By Lemma 4, if a multi-agent system is controllable, the associated interconnection graph is leader-follower connected. Since, by Definition 3, an undirected interconnection graph with single leader and leader-follower connectedness reduces to a connected one (and a weakly connected graph for a directed interconnection graph), it follows that the interconnection graph associated with a controllable multi-agent system under single leader is connected (and weakly connected in case of a directed graph). Then, by Lemma 2, the interconnection graph associated with a controllable multi-agent system under single leader has a spanning tree. By Lemma 1, the controllable multi-agent system with a single leader solves a consensus problem.

Concerning multiple leaders, it can be readily seen from Definition 3 that the interconnection graph with multiple leaders and leader-follower connectedness need not be connected. Accordingly, by Lemma 4, the interconnection graph associated with a controllable multi-agent system under multiple leaders may not be connected. Then, by Lemma 1, the controllable multi-agent system with multiple leaders may not solve a consensus problem.

Part (ii) Suppose system (1) has a single leader and cannot solve a consensus problem, it follows from Lemma 1 that there is no spanning tree in the interconnection graph. Accordingly, by Lemma 3, the Laplacian matrix L has at least two zero eigenvalues. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N+1}$ be the eigenvalues of L_f and L, respectively. Then $\lambda_N = \lambda_{N+1} = 0$. Since, by Lemma 5, $\lambda_N \geq \mu_N \geq \lambda_{N+1}$, one has $\mu_N = 0$. That is, L_f and L share a common eigenvalue zero.

Part (iii) Since whether an interconnection graph has a spanning tree is not affected by the selecting of weights among agents, the consensus property is invariant under different weights; while the controllability is a property affected by the selecting of different weights, which is illustrated in the following Example 2. ■

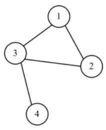


Fig. 2. An undirected interconnection graph with agent 4 being the leader.

Example 2: Concerning the interconnection graph indicated in Fig.2, the corresponding multi-agent system is uncontrollable since L and L_f share a common eigenvalue 3. If the weighted matrix is chosen as

$$A = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 3 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{bmatrix}.$$

It can be seen that in this case L and L_f share no common eigenvalues; and then by Lemma 2.2 in [2], the multi-agent system is turned into controllable.

Example 3: The example is employed to show that although the multi-agent system is controllable, it may not solve a consensus problem when the leaders are multiple. Consider Fig.3, computations show that the corresponding Laplacian matrix L and the multi-agent system matrix L_f share no common eigenvalues. By Lemma 2.2 in [2], the multi-agent system is controllable. However, it cannot solve a consensus problem since the interconnection graph has no spanning tree.

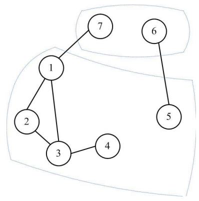


Fig. 3. An undirected interconnection graph with agents 6 and 7 taking the leader role.

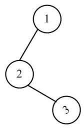
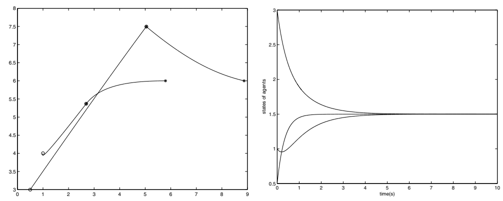


Fig. 4. An undirected graph with agent 3 taking the leader role.

Example 4: Let $\mathcal{G}$ be an interconnection graph indicated in Fig.4. Calculations show that L and L_f share no common eigenvalues, and accordingly the system is controllable. Since there is a spanning tree in the graph, the system also solves a consensus problem. The controlled trajectories of the two followers v_2, v_3 are shown in (a) of Fig.5, where the final configuration is a straight line; and the consensus trajectories of the three agents are shown in (b) of Fig.5.

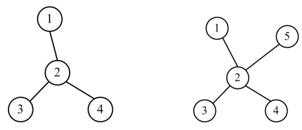


(a) (b)

Fig. 5. (a) The controllable trajectories of the two followers v_1, v_2; (b) The consensus trajectories of the three agents.

V. CHANGING AN UNCONTROLLABLE SYSTEM INTO A CONTROLLABLE ONE

In this section, we focus on turning a kind of uncontrollable tree into a controllable one by selecting different weights between agents. We first consider a related example.



(a) (b)

Fig. 6. (a) A tree with 4 nodes; (b) A tree with 5 nodes

Calculations show that the multi-agent systems associated with the trees in Fig.6 are always uncontrollable no matter how the single leader is selected. So if we want to change the systems with a single leader to be controllable, appropriate weights for communication links ought to be assigned.

Lemma 6: [11] If the multi-agent system (3) is controllable, none of two followers have the same neighbor set of followers.

Example 5: Associated with (a) of Fig.6, the system can be changed to a controllable one by assigning weight 2 between agent 1 and agent 2 and appointing agent 4 to take the leader role. For (b) of Fig.6, the system can be turned to be controllable by assigning weight 2 between agent 1 and agent 2, weight 3 between agent 2 and agent 3, and appointing agent 5 to be the leader.

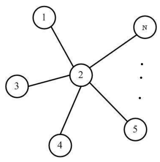


Fig. 7. An interconnection graph corresponding to N agents.

Inspired by this example, we give the following result.

Theorem 3: A multi-agent system with the interconnection graph indicated in Fig.7 is uncontrollable no matter how the single leader is selected. However, it can be turned to be a controllable one by selecting appropriate weights for some communication links according to the following steps:

(i) Choose N to take the leader role,
(ii) Select different weights for different links except that between agent 2 and agent N.

Proof: If agent 2 is taken to play the leader role, it follows from the interconnection graph topology structure indicated in Fig. 7 that all the followers are the neighbors of the leader. Direct computations show that $l_{fl} = L_f l_{fl}$. Accordingly, the controllability matrix C is rank defect. The multi-agent system is then uncontrollable.

If we select, say agent 1, to take the leader role. From the interconnection graph in Fig. 7, followers $3, 4, \cdots, N$ have the same set of follower neighbor set, i.e., $\{2\}$. By Lemma 6, the system is uncontrollable. The same arguments can be repeated for agents $3, 4, \cdots, N$ if one of them is chosen to play the leader role. The above reasonings imply that the multi-agent system with the interconnection graph indicated in Fig.7 is uncontrollable no matter how the single leader is selected.

Next, we show the second part of the result. For the convenience of statement, we assign different weights for different links according to (ii) for a special case, say, assigning weight 2 between nodes 2 and 1, weight 3 between 2 and 3, $\cdots$, and weight $N-2$ between 2 and $N-2$. The general case can be proved in the same way. The weights between nodes 2 and N, 2 and $N-1$ are invariant, i.e., still being 1. In this way, the corresponding system matrices are

$$L_f = \begin{bmatrix} 2 & -2 & 0 & \cdots & 0 & 0 \\ -2, & \frac{N^2-3N+4}{2}, & -3, & \cdots & -(N-2), & -1 \\ 0 & -3 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -(N-2) & 0 & \cdots & N-2 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$l_{fl} = \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 & 0 \end{bmatrix}^T.$$

By computing, the controllable matrix is

$$C = \begin{bmatrix} 0 & 2 & 2x_1+2^2 & \cdots & a \\ -1 & -x_1 & -x_2 & \cdots & -x_{N-2} \\ 0 & 3 & 3x_1+3^2 & \cdots & b \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & N-2, & t & \cdots & c \\ 0 & 1 & x_1+1 & \cdots & d \end{bmatrix}$$

where

$$\begin{aligned} a =& 2^{(N-3)}x_1 + 2^{(N-4)}x_2 + \cdots + 2x_{(N-3)} + 2^{(N-2)} \\ b =& 3^{(N-3)}x_1 + 3^{(N-4)}x_2 + \cdots + 3x_{(N-3)} + 3^{(N-2)} \\ c =& (N-2)^{(N-3)}x_1 + (N-2)^{(N-4)}x_2 + \cdots \\ &+ (N-2)x_{(N-3)} + (N-2)^{(N-2)} \\ d =& x_1 + x_2 + \cdots + x_{N-3} + 1 \\ t =& (N-2)x_1 + (N-2)^2 \\ x_1 =& \frac{N^2-3N}{2} + 2 \\ x_2 =& 1^2 + 2^2 + \cdots + (N-2)^2 + x_1^2 \\ =& \frac{(N-2)(N-1)(2N-3)}{6} + x_1^2 \\ &\vdots \\ x_{N-2} =& [1 + 2^{(N-2)} + \cdots + (N-2)^{(N-2)}] \\ &+ [1 + 2^{(N-3)} + \cdots + (N-2)^{(N-3)}]x_1 \\ &+ [1^{(N-4)} + 2^{(N-4)} + \cdots + (N-2)^{(N-4)}]x_2 \\ &+ \cdots \\ &+ [1^2 + 2^2 + \cdots + (N-2)^2]x_{N-4} + x_1 x_{N-3} \end{aligned}$$

By elementary transformations, C can be transformed to the following matrix M, with

$$M = \begin{bmatrix} f & G \\ -1 & h^T \end{bmatrix}, \quad (8)$$

where $f = 0_{(N-2)\times 1}$, $G \in R^{(N-2)\times(N-2)}$, $h^T \in R^{1\times(N-2)}$, and

$$h^T = \begin{bmatrix} -x_1, & -x_2, & -x_3, & -x_4, & \cdots, & -x_{N-2} \end{bmatrix},$$

$$G = \begin{bmatrix} 1 & x_1+1 & \cdots & d \\ 2 & 2x_1+2^2 & \cdots & a \\ \vdots & \vdots & \vdots & \vdots \\ N-2 & (N-2)x_1+(N-2)^2 & \cdots & c \end{bmatrix}.$$

Applying elementary transformations to G, one has

$$G \sim \begin{bmatrix} 1 & x_1+1 & \cdots & d \\ 0 & 2^2-2 & \cdots & e \\ \vdots & \vdots & \vdots & \vdots \\ 0 & (N-2)^2-(N-2) & \cdots & f \end{bmatrix}$$
$$\sim \begin{bmatrix} h_1 & * & \cdots & * \\ & h_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & h_{N-2} \end{bmatrix},$$

where '$\sim$' stands for the equivalence of two matrices in the sense of elementary transformations,

$$\begin{aligned} e =& [2^{(N-3)}-2]x_1 + [2^{(N-4)}-2]x_2 + \cdots \\ &+ (2^2-2)x_{N-4} + 2^{N-2} - 2 \\ f =& [(N-2)^{(N-3)} - (N-2)]x_1 \\ &+ [(N-2)^{(N-4)} - (N-2)]x_2 + \cdots \\ &+ [(N-2)^2 - (N-2)]x_{N-4} \\ &+ [(N-2)^{N-2} - (N-2)] \\ h_1 =& 1, h_2 = 2^2 - 2, \end{aligned}$$

and $h_3, \cdots, h_{N-2}$ are expressed iteratively as follows: For $i = 2, \cdots, N-3$,

$$\begin{aligned} h_{i+1} =& (i+1)^{i+1} - (i+1) - \frac{h_{i+1,2}}{h_2}\tilde{h}_{2,i+1} \\ &- \frac{h_{i+1,3}}{h_3}\tilde{h}_{3,i+1} - \cdots - \frac{h_{i+1,i}}{h_i}\tilde{h}_{i,i+1}, \end{aligned}$$

with

$$\begin{aligned} h_{i+1,k} \triangleq& (i+1)^{i+1} - (i+1) - \frac{h_{i+1,2}}{h_2}\tilde{h}_{2,i+1} - \cdots \\ &- \frac{h_{i+1,k-1}}{h_{k-1}}, \qquad k = 2, \cdots, i; \end{aligned}$$

and $\tilde{h}_{2,i+1} = 2^{i+1} - 2$, $\tilde{h}_{k+1,i+1} = \hat{h}_{k+1,i+1}\tilde{h}_{k,i+1}$, $k = 2, \cdots, i-1$; where $\hat{h}_{k+1,i+1}$ is defined as below: Let

$$\hat{h}_k \triangleq k^k - k - \frac{h_{k,2}}{h_2}\tilde{h}_{2,k} - \frac{h_{k,3}}{h_3}\tilde{h}_{3,k} - \cdots - \frac{h_{k,k-1}}{h_{k-1}},$$

that is, $\hat{h}_k$ is obtained by deleting the last multiplying term of h_k, where

$$h_k = k^k - k - \frac{h_{k,2}}{h_2}\tilde{h}_{2,k} - \frac{h_{k,3}}{h_3}\tilde{h}_{3,k} - \cdots - \frac{h_{k,k-1}}{h_{k-1}}\tilde{h}_{k-1,k}.$$

Then

$$\hat{h}_{k+1,i+1} \triangleq * - \frac{*}{h_2} - \cdots - \frac{*}{h_{k-1}},$$

where $*$'s indicate the terms obtained via $\hat{h}_k$ by replacing the corresponding numbers of $\hat{h}_k$ with $k+1$ except those power exponents numbers.

In this way, the values of $h_3, \cdots, h_{N-2}$ can be written as

$$\begin{aligned} h_3 =& 3^3 - 3 - \frac{3^2-3}{2^2-2}(2^3-2) \\ =& 3^3 - 3 - \frac{h_{3,2}}{h_2}\tilde{h}_{(2,3)}, \\ &\vdots \\ h_{N-3} =& (N-3)^{(N-3)} - (N-3) - \frac{h_{N-3,2}}{h_2}\tilde{h}_{(2,N-3)} \\ &- \frac{h_{N-3,3}}{h_3}\tilde{h}_{(3,N-3)} - \cdots - \frac{h_{N-3,N-5}}{h_{N-5}}\tilde{h}_{(N-5,N-3)} \\ &- \frac{h_{N-3,N-4}}{h_{N-4}}\tilde{h}_{(N-4,N-3)} \\ h_{N-2} =& (N-2)^{(N-2)} - (N-2) - \frac{h_{N-2,2}}{h_2}\tilde{h}_{(2,N-2)} \\ &- \frac{h_{N-2,3}}{h_3}\tilde{h}_{(3,N-2)} - \cdots - \frac{h_{N-2,N-4}}{h_{N-4}}\tilde{h}_{(N-4,N-2)} \\ &- \frac{h_{N-2,N-3}}{h_{N-3}}\tilde{h}_{(N-3,N-2)} \end{aligned}$$

Since $h_1 > 0$ and h_i is increasing, G is full rank. This, together with the specific structure of M expressed in (8), gives rise to that M is nonsingular. As a consequence, C is nonsingular, i.e., the system is controllable. ■

VI. CONCLUSIONS

We have studied the controllability problem of multi-agent systems under directed and undirected interconnection topologies. The controllability characterizations are presented with respect to the proposed interconnection graphs. Some connections between controllability and consensus problem are revealed. In the results, the corresponding selection of leaders is also given. The work provides insight into the role that the interconnection graph plays in the controllability problem of multi-agent systems.

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