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Downer and Perron Branches in Interconnection Topologies for Coordination and Control of Multi-agent Networks*

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***Abstract*— In this paper, we devote to the study of interconnection topologies for the coordinated control of multi-agent networks. It turns out that downer and Perron branches contribute to the understanding of coordinated behavior of multiple agents from the view point of interconnection topology structures. In particular, we show that the uncontrollability of multi-agent systems is equivalent to the existence of a downer branch when the interconnection graph is a tree. For general interconnection graph, it is shown that the existence of a Perron branch leads to the uncontrollability of the system in most cases. In the latter case, two equivalent conditions are also given. When there are edge failures occurred in the graph, a result is presented to cope with the robustness of the controllability. In all the results, the selection of leaders is outlined.**

***Index Terms*— Multi-agent networks. Multi-agent controllability. Interconnection topology.**

I. Introduction

Recent years have seen an increased interest in the study of coordination and control of multi-agent systems, see e.g. [1], [7], [22]. The study is inspired by numerous applications of such networks in various fields including, e.g. the cooperative control and coordination of multiple robots and unmanned aerial vehicles, and the investigation of the problems emerged from the swarming behavior of biological systems, such as flocks of birds, colonies of bacteria, etc., where relatively simple creatures can collectively perform complex, meaningful, and intelligent tasks.

With nodes representing dynamic units and links indicating the interconnections between them, graph theory proves to be a natural framework for modeling and treatment of networks of dynamic agents. The characterization of properties of such systems then relies closely on the structure and interconnection topology of the graph/network. This motivates in the paper the study of interconnection topology structures for the coordination control of multi-agent systems. It is known that the study of controllability is the core of classical control, playing a fundamental role in analysis and synthesis of linear control systems. For multi-agent systems, the current version of controllability was put forward for the first time by Tanner in [23], where sufficient and necessary algebraic conditions were derived. He also pointed out that the lack of a graph theoretic characterization of the controllability property prevents controllable interconnection topologies from building. This motivated subsequent analysis of connections between controllability and interconnection topology structures. In [20], [8], the authors proved that symmetric structure leads to uncontrollability. In [15], [10], results were derived for controllability under switching topology and time-delay and in [11], [9] uncontrollable topology structures and graph theoretic properties were revealed. Recently, some other related results were also reported in e.g. [17], [16], [24], [12], [18], [19].

The contribution of the paper includes the revealment of the equivalence between uncontrollability and a downer branch for tree topologies, as well as a relationship between Perron branches and uncontrollability of the system for general interconnection graph. For the latter case, two equivalent conditions are also given, which are, respectively, based on Fiedler vector and algebraic expression. In addition to these developments, a result on robust controllability is presented when there are edge failures occurred in the graph.

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II. Preliminaries and problem formulation

A. Graph preliminaries and notations

An undirected graph $\mathcal{G}$ consists of a node set $\mathcal{V}$ and an edge set $\mathcal{E} = \{(v_i, v_j)|v_i, v_j \in \mathcal{V}, i \neq j\}$. v_i and v_j are neighbors if $(v_i, v_j) \in \mathcal{E}$. The number of neighbors of v_i is its degree, denoted by d_i. A path $v_{i_0}v_{i_1}\cdots v_{i_s}$ is a finite sequence of nodes such that $v_{i_{k-1}}$ and v_{i_k} are neighbors $k = 1, \cdots, s$. A graph $\mathcal{G}$ is connected if there is a path between any pair of distinct nodes. For two graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E}), \mathcal{G}' = (\mathcal{V}', \mathcal{E}')$, we call $\mathcal{G}'$ a subgraph of $\mathcal{G}$, denoted by $\mathcal{G}' \subseteq \mathcal{G}$, if $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$. A subgraph $\mathcal{G}'$ is said to be induced from $\mathcal{G}$ if it is obtained by deleting a subset of nodes and all the edges connecting to those nodes. An induced subgraph of an undirected graph, which is maximal and connected, is said to be a connected component of the graph.

Throughout the paper, all graphs are assumed to be simple, i.e., they have no multiple edges and loops. By $\mathcal{L}(\mathcal{G})$ (or simply, $\mathcal{L}$), we mean the Laplacian matrix of $\mathcal{G}$, defined by

$\mathcal{L} = D - A$, where D is the diagonal degree matrix of $\mathcal{G}$ and A is the adjacency matrix of $\mathcal{G}$. Since $\mathcal{L}$ is a positive semidefinite matrix with the smallest eigenvalue 0, it can be assumed that the eigenvalues of $\mathcal{L}$ are $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. The second smallest eigenvalue of $\mathcal{L}$, denoted by $a(\mathcal{G})$, is said to be the algebraic connectivity of $\mathcal{G}$. Fiedler showed that $a(\mathcal{G})$ is 0 if and only if $\mathcal{G}$ is disconnected [6]. The eigenvectors corresponding to $a(\mathcal{G})$ are called Fiedler vectors of the graph $\mathcal{G}$. For a vector Y, the notation $Y(v)$ represents the coordinate of Y corresponding to the vertex v. With respect to an eigenvector Y of $\mathcal{L}$, a vertex v is said to be nonzero (zero, negative, positive) if $Y(v) \neq 0$ ($Y(v) = 0, Y(v) < 0, Y(v) > 0$, respectively). A subgraph $\mathcal{H}$ of $\mathcal{G}$ containing a nonzero vertex of $\mathcal{G}$ is called a nonzero subgraph of $\mathcal{G}$. A subgraph $\mathcal{H}$ of $\mathcal{G}$ is called positive if each vertex of $\mathcal{H}$ is positive. Denote by $\tau(B)$ the smallest eigenvalue of a square symmetric matrix B.

Let $\mathcal{W}$ be a set of vertices and edges in $\mathcal{G}$, we denote by $\mathcal{G} \setminus \mathcal{W}$ the graph obtained by deleting all the elements of $\mathcal{W}$ from $\mathcal{G}$. It is understood that when a vertex is deleted, all edges incident with it are deleted as well, but when an edge is deleted, the vertices incident with it are not. By a branch at vertex set $\mathcal{W}$ of $\mathcal{G}$ we mean a component of $\mathcal{G} \setminus \mathcal{W}$. A vertex v of $\mathcal{G}$ is a point of articulation (or cutpoint) if $\mathcal{G} \setminus v$, the graph formed by removing v and its incident edges, is disconnected. A matrix $A = (a_{ij})$ is called an M-matrix if $a_{ij} \leq 0$ whenever $i \neq j$ and all principal minors of A are positive.

B. Problem formulation

Consider the multi-agent system given by

$$\begin{cases} \dot{x}_i = u_i, & i = 1, \ldots, N \\ \dot{x}_{N+j} = u_{N+j}, & j = 1, \ldots, l \end{cases} \tag{1}$$

where N and l represent the number of followers and leaders, respectively; and x_i indicates the state of the ith agent, $i = 1, \cdots, N + l$.

In [23], the interconnection graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ is introduced, which is an undirected graph consisting of a set of nodes, $\mathcal{V} = \{v_1, \ldots, v_N, v_{N+1}, \ldots, v_{N+l}\}$, indexed by the agents in the group; and a set of edges, $\mathcal{E} = \{(v_i, v_j) \in \mathcal{V} \times \mathcal{V} | \, v_i \sim v_j\}$, containing unordered pairs of nodes that correspond to interconnected agents.

In [11], the interconnection graph $\mathcal{G}$ is assumed to be connected. The controllability problem can be studied under this assumption. The topology of an interconnection graph $\mathcal{G}$ is said to be fixed if each node of $\mathcal{G}$ has a fixed neighbor set. Let $\mathcal{N}_i = \{j \,|\, v_i \sim v_j; j \neq i\}$, which is the neighboring set of v_i; and define the protocol as follows:

$$u_i = -\sum_{j \in \mathcal{N}_i} (x_i - x_j). \tag{2}$$

Take $x_{N+1}, \cdots, x_{N+l}$ to play leaders role, and rename the agents as $y_i \overset{\Delta}{=} x_i, i = 1, \ldots, N$; $z_j \overset{\Delta}{=} x_{N+j}, j = 1, \ldots, l$. Let y, z and u denote the stack vectors of all y_i, z_j, and u_{N+j}, respectively, $i = 1, \cdots, N$; $j = 1, \ldots, l$. In this leader-follower framework, the interconnections with the leaders are assumed to be unidirectional, that is, the leaders' neighbors still obey (2), but the leaders are free of such a constraint and are allowed to pick u_{N+j} arbitrarily, $j = 1, \ldots, l$. Then, under protocol (2), the multi-agent system (1) reads

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = - \begin{bmatrix} \mathcal{F} & \mathcal{R} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix},$$

where $\mathcal{F}$ is the matrix obtained from the Laplacian $\mathcal{L}$ of $\mathcal{G}$ after deleting the last l rows and columns. $\mathcal{R}$ is the $N \times l$ submatrix consisting of the first N elements of the deleted columns. The dynamics of the followers corresponding to the y component of the equation is extracted as

$$\dot{y} = -\mathcal{F}y - \mathcal{R}z. \tag{3}$$

Definition 1: The multi-agent system (1) is said to be controllable under leaders $x_{N+j}, j = 1, \ldots, l$, and fixed topology if system (3) is controllable under control input z.

III. Controllability and topology structures

A. Robust controllability and the same neighbor set

One kind of communication uncertainty can be measured by the node or edge failures of the interconnection graph. In what follows, we consider the controllability problem with respect to edge failures occurred in the graph which contains a number of vertices with the same neighbor set. That is, we consider how about the controllability/uncontrollability if some edges are added or deleted in the interconnection graph.

The following lemma plays an important role in the development of the results.

Lemma 1: [11] The multi-agent system with (undirected) weighted interconnection graphs is controllable if and only if there is no eigenvector of Laplacian matrix $\mathcal{L}$ taking 0 on the elements corresponding to the leaders.

Definition 2: The κ nodes $v_{i_1}, \cdots, v_{i_\kappa}$ in the graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ are said to have the same neighbor set if each of these nodes has the same set of neighbors $\{v_{i_{\kappa+1}}, \cdots, v_{i_{\kappa+\varrho}}\}$, where $v_{i_j} \in \mathcal{V}$, $i_h \neq i_j$ for $\forall h \neq j$.

The concept of the same neighbor set is meaningless for a single node case, i.e., $\kappa = 1$. So $\kappa \geq 2$.

Proposition 1: [11] The multi-agent system (1) is uncontrollable if the following two conditions are fulfilled simultaneously:

(i) there are nodes with the same neighbor set in the interconnection graph $\mathcal{G}$;

(ii) leaders are selected as follows:

- when $\kappa = 2$, i.e., there are only two nodes with the same neighbor set, the leaders are to be selected from the remaining nodes in $\mathcal{G}$ other than the two nodes with the same neighbor set.

- when $\kappa \geq 3$, the number of leaders is not greater than $\kappa - 2$ and the leaders are to be selected arbitrarily.

Lemma 2: (Lemma 2.1 of [5]) Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be a graph with vertex subset $\mathcal{V}' = \{v_1, \cdots, v_\kappa\}$ having the same set of neighbors $\{v_{\kappa+1}, \cdots, v_{\kappa+\varrho}\}$, where $\mathcal{V} = \{v_1, \cdots, v_\kappa, \cdots, v_{\kappa+\varrho}, \cdots, v_n\}$. Then the Laplacian matrix of the graph $\mathcal{G}$ has at least $\kappa - 1$ equal eigenvalues and they are all equal to the cardinality ϱ of the neighbor set. Also the corresponding $\kappa - 1$ eigenvectors are $[1, -1, 0, \cdots, 0]^T$, $[1, 0, -1, 0, \cdots, 0]^T, \cdots, [\underbrace{1, 0, \cdots, -1}_{\kappa}, 0, \cdots, 0]^T$.

Lemma 3: (Theorem 2.2 of [21]) Suppose i and j are fixed but arbitrary nonadjacent vertices of $\mathcal{G}$. Let $\mathcal{G}^+ = \mathcal{G} + \{i, j\}$ be the graph obtained from $\mathcal{G}$ by adding an edge $\{i, j\}$. Then $N(i) = N(j)$ if and only if the spectrum of $\mathcal{L}(\mathcal{G}^+)$ overlaps the spectrum of $\mathcal{L}(\mathcal{G})$ in $n - 1$ places.

Remark 1: The result inspires the following considerations:

- It follows from the proof of Lemma 3 that $N(i) = N(j)$ if and only if $\mathcal{L}(\mathcal{G})K = K\mathcal{L}(\mathcal{G})$ holds not only for the case of two vertices with the same neighbor set but also holds for the general case when vertices have the same neighbor set. Note that the same neighbor set provides a graph theoretical characterization for uncontrollability in our recent work [11]. Then $\mathcal{L}(\mathcal{G})K = K\mathcal{L}(\mathcal{G})$ presents an algebraic characterization for the same neighbor set, which also applies to the general case.
- The result relates the same neighbor set to the spectrum of $\mathcal{L}(\mathcal{G}^+)$ and $\mathcal{L}(\mathcal{G})$. This may bring some ideas for the understanding of controllability problem (see e.g., the following Theorem 1).

Lemma 4: (Theorem 3.3 of [4]) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with a vertex subset $\mathcal{V}' = \{v_1, v_2, \cdots, v_\kappa\}$ having the same set of neighbors $\{v_{\kappa+1}, \cdots, v_{\kappa+\varrho}\}$, where $\mathcal{V} = \{v_1, \cdots, v_\kappa, \cdots, v_{\kappa+\varrho}, \cdots, v_n\}$. Also, let $\mathcal{E}^+ = \mathcal{E} \cup \mathcal{E}'$, where $\mathcal{E}' \subseteq \mathcal{V}' \times \mathcal{V}'$. If $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ has eigenvalues $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_\kappa = 0$, then the eigenvalues of $\mathcal{L}(\mathcal{G}^+)$, where $\mathcal{G}^+ = (\mathcal{V}, \mathcal{E}^+)$, are as follows: those eigenvalues of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, which are equal to ϱ ($\kappa - 1$ in number), are incremented by $a_i, i = 1, \cdots, \kappa - 1$, and the remaining eigenvalues are same.

Theorem 1: Let $\mathcal{G}$ be a graph with a vertex subset $\mathcal{V}' = \{v_1, \cdots, v_\kappa\}$ having the same set of neighbors. Denote by $\mathcal{G}^+$ an interconnection graph which is obtained from $\mathcal{G}$ by adding edges between vertices in $\mathcal{V}'$, and let $\mathcal{G}^{+-}$ be the graph obtained from $\mathcal{G}$ by adding or deleting edges between vertices in $\mathcal{V} \setminus \mathcal{V}'$. Then, the following assertions hold:

(i) A multi-agent system with interconnection graph $\mathcal{G}^+$ is uncontrollable if leaders are selected from $\mathcal{V} \setminus \mathcal{V}'$.

(ii) In case $\kappa \geq 4$ and $\mathcal{G}^+ = \mathcal{G} + \{i, j\}$, where $v_i, v_j \in \mathcal{V}'$, i.e., $\mathcal{G}^+$ is the graph obtained from $\mathcal{G}$ by adding a single edge $\{i, j\}$, then the system with interconnection graph $\mathcal{G}^+$ is uncontrollable under any single leader.

(iii) Proposition 1 still holds for a multi-agent system with interconnection graph $\mathcal{G}^{+-}$, as well as for a multi-agent system with interconnection graph $\mathcal{G}^+ = \mathcal{G} + \{i, j\}$ in which there are $\kappa - 2$ vertices with the same neighbor set, where $v_i, v_j \in \mathcal{V}'$.

Proof (i) It follows from the proof of Lemma 4 in [4] that if $x_i' = [x_{i,1}, \cdots, x_{i,\kappa}]^T$ is an eigenvector of $\mathcal{G}'$ associated with the eigenvalue a_i, the corresponding $x_i = [x_{i,1}, \cdots, x_{i,\kappa}, \underbrace{0, \cdots, 0}_{n-\kappa}]^T$ is an eigenvector of $\mathcal{G}^+$ associated with the eigenvalue $\varrho + a_i, i = 1, \cdots, \kappa - 1$. This, together with Lemma 1, yields the conclusion.

(ii) By Lemma 2, ϱ is an eigenvalue of $\mathcal{G}$ with multiplicity $\kappa - 1$. Lemma 3 then implies that ϱ is an eigenvalue of $\mathcal{G}^+$ with multiplicity at least $\kappa - 2$. Accordingly, κ is a multiple eigenvalue of $\mathcal{G}^+$ since $\kappa \geq 4$. Then $\mathcal{L}(\mathcal{G})^+$ has two linearly independent eigenvectors. With respect to any fixed coordinate i, an eigenvector of $\mathcal{L}(\mathcal{G})^+$ can be generated by using a linear combination of these two linearly independent eigenvectors. Combining this with Lemma 1 gives rise to the assertion that the system with interconnection graph $\mathcal{G}^+$ is uncontrollable under any single leader.

(iii) The first assertion of (iii) follows from the observation that the same neighbor set is unaffected by adding or deleting edges between vertices in $\mathcal{V} \setminus \mathcal{V}'$. The second assertion is true because after the connection of vertices v_i, v_j with a single edge $\{i, j\}$, the remaining $\kappa - 2$ vertices in $\mathcal{V}'$ have the same neighbor set in $\mathcal{G} + \{i, j\}$.

B. Perron branch, Fiedler vectors and controllability

Lemma 1 implies that controllability closely relates to the eigenvectors of Laplacian matrix. This inspires the investigation of controllability through Laplacian eigenvectors. In particular, we study controllability in this subsection by considering Fiedler vectors of a connected graph.

A Perron branch at vertex set $\mathcal{S}$ is a connected component of $\mathcal{G} \setminus \mathcal{S}$ with the smallest eigenvalue of the corresponding principle submatrix of $\mathcal{L}(\mathcal{G})$ less than or equal to $a(\mathcal{G})$.

Lemma 5: [2] Let $\mathcal{G}$ be a connected simple graph and $a(\mathcal{G})$ the algebraic connectivity. Let $\mathcal{W}$ be a set of vertices of $\mathcal{G}$ such that $\mathcal{G} \setminus \mathcal{W}$ is disconnected. Let $\mathcal{G}_1, \mathcal{G}_2$ be two components of $\mathcal{G} \setminus \mathcal{W}$ and let $\mathcal{L}_1, \mathcal{L}_2$ be the principal submatrices of $\mathcal{L}$ corresponding to $\mathcal{G}_1, \mathcal{G}_2$, respectively. Suppose $\tau(\mathcal{L}_1) \leq \tau(\mathcal{L}_2)$. Then either $\tau(\mathcal{L}_2) > a(\mathcal{G})$ or $\tau(\mathcal{L}_1) = \tau(\mathcal{L}_2) = a(\mathcal{G})$.

Lemma 6: [2] Let $\mathcal{G}$ be a connected graph and Y be a Fiedler vector. Let $\mathcal{W}$ be a nonempty set of vertices of $\mathcal{G}$ such that $Y(u) = 0$ for all $u \in \mathcal{W}$ and suppose $\mathcal{G} \setminus \mathcal{W}$ is disconnected with at least two nonzero components, $\mathcal{G}_1$ and $\mathcal{G}_2$. Let $\mathcal{L}_i$ and Y_i be the principal submatrix of $\mathcal{L}$ and the subvector of Y corresponding to $\mathcal{G}_i, i = 1, 2$. Then $\tau(\mathcal{L}_1) = \tau(\mathcal{L}_2) = a(\mathcal{G})$.

Lemma 7: [20] The multi-agent system is controllable if and only if none of the eigenvectors of $\mathcal{F}$ is (simultaneously) orthogonal to (all columns of) $\mathcal{R}$. Moreover, if $\mathcal{F}$ does not have distinct eigenvalues, then the system is uncontrollable.

A follower subgraph $\mathcal{G}_f$ of the interconnection graph $\mathcal{G}$ is the subgraph induced by the follower set $\mathcal{V}_f$. Similarly, A leader subgraph $\mathcal{G}_l$ is the one induced by the leader set $\mathcal{V}_l$. Assume that $\mathcal{G}_f$ consists of γ connected components $\mathcal{G}_{c_1}, \ldots, \mathcal{G}_{c_\gamma}$, with $\{v_1, \ldots, v_{n_1}\}$, $\{v_{n_1+1}, \ldots, v_{n_2}\}$, $\ldots$, and $\{v_{n_{\gamma-1}+1}, \ldots, v_N\}$ being their node sets, respectively. Denote by $\mathcal{L}[i_1, \cdots, i_m]$ the principal submatrix obtained by selecting the i_1th, $\ldots$, i_mth rows and columns of $\mathcal{L}$. Then $\mathcal{L}[n_{i-1}+1, \cdots, n_i]$ corresponds to $\mathcal{G}_{c_i}$, where $n_0 = 0, n_\gamma = N$, $i = 1, \cdots, \gamma$. The following assertion is a combination of Lemmas 1, 2 in [9].

Lemma 8: $\mathcal{L}[1, \cdots, N]$ is positive definite and $\mathcal{L}[1, \cdots, N] = diag\{\mathcal{L}[1, \cdots, n_1], \mathcal{L}[n_1+1, \cdots, n_2], \cdots, \mathcal{L}[n_{\gamma-1}+1, \cdots, N]\}$, where $\mathcal{L}[1, \cdots, n_1], \cdots, \mathcal{L}[n_{\gamma-1}+1, \cdots, N]$ are all positive definite submatrices too.

Denote by $\mathcal{V}$ and $\mathcal{V}_{c_i}$ the vertex set of $\mathcal{G}$ and $\mathcal{G}_{c_i}$, respectively. Let $\mathcal{V} \setminus \{\mathcal{V}_{c_i}, \mathcal{V}_{c_j}\}$ represent the vertex set $\mathcal{V}$ except those of $\mathcal{V}_{c_i}$ and $\mathcal{V}_{c_j}$, where $i, j \in \{1, \cdots, \gamma\}$.

Theorem 2: For a multi-agent system with interconnection graph $\mathcal{G}$. The following statements are equivalent:

(i) there are two principal submatrices $\mathcal{L}[n_{i-1}+1, \cdots, n_i]$ and $\mathcal{L}[n_{j-1}+1, \cdots, n_j]$ with $\tau(\mathcal{L}[n_{j-1}+1, \cdots, n_j]) \leq a(\mathcal{G})$, where $\tau(\mathcal{L}[n_{i-1}+1, \cdots, n_i]) \leq \tau(\mathcal{L}[n_{j-1}+1, \cdots, n_j])$, $i \neq j$, $i, j \in \{1, \cdots, \gamma\}$;

(ii) $\mathcal{G}_{c_i}$ and $\mathcal{G}_{c_j}$ are Perron branches with the smallest eigenvalue of the corresponding principal submatrix of $\mathcal{L}$ equal to $a(\mathcal{G})$;

(iii) there is a Fiedler vector with the coordinates corresponding to the leader vertices in $\mathcal{V}_l$ being zero and there are two nonzero components in $\mathcal{G} \setminus \mathcal{G}_l$.

Moreover, the following assertions hold:

(a) the system is uncontrollable if any of the above conditions (i)-(iii) are satisfied, with no matter how leaders are selected from the vertex set $\mathcal{V} \setminus \{\mathcal{V}_{c_i}, \mathcal{V}_{c_j}\}$;

(b) the system is uncontrollable if there exists some $\mathcal{G}_{c_\psi}$, which is a Perron branch with the smallest eigenvalue of the corresponding principal submatrix of $\mathcal{L}$ equal to $a(\mathcal{G})$, where $\psi \geq 2$. In this case, the system is uncontrollable with no matter how leaders selected from $\mathcal{V} \setminus \{\mathcal{V}_{c_i}, \mathcal{V}_{c_j}\}$, where arbitrary $i, j \in \{1, \cdots, \psi\}, i \neq j$.

Proof The result will be proved according to the following steps: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). The uncontrollability of the system will be shown in the proof procedure.

Denote $\mathcal{F}_i \overset{\Delta}{=} \mathcal{L}[n_{i-1}+1, \cdots, n_i]$. Then, by Lemma 8, $\mathcal{F} = diag\{\mathcal{F}_1, \cdots, \mathcal{F}_m\}$. Since $\tau(\mathcal{F}_i) \leq \tau(\mathcal{F}_j)$ and $\tau(\mathcal{F}_j) \leq a(\mathcal{G})$, it follows from Lemma 5 that $\tau(\mathcal{F}_i) = \tau(\mathcal{F}_j) = a(\mathcal{G})$. As a consequence, the two components $\mathcal{G}_{c_i}$ and $\mathcal{G}_{c_j}$ that correspond to $\mathcal{F}_i$ and $\mathcal{F}_j$, respectively, are both Perron branches and one can get from Lemma 7 that the system is uncontrollable. In the sequel, in addition to the proof of (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), we will also point out how to select leaders so that the system is uncontrollable.

Since the interconnection graph $\mathcal{G}$ is connected, Lemma 8 tells us that both $\mathcal{F}_i$ and $\mathcal{F}_j$ are nonsingular symmetric M-matrices and then its inverse is positive. Let Z_i and Z_j be the eigenvector of $\mathcal{F}_i$ and $\mathcal{F}_j$ associated with $a(\mathcal{G})$, respectively. It follows from the Perron-Frobenius theorem that both Z_i and Z_j are positive. Then the same lines of proof for the second assertion of Theorem 9 in [3] can be repeated to show that $X_{ij} = [0^T, \cdots, 0^T, Z_i^T, 0^T, \cdots, 0^T, -kZ_j^T, 0^T, \cdots, 0^T]^T$ is a Fiedler vector of the interconnection graph $\mathcal{G}$, where $k = \frac{\sum_{v \in \mathcal{G}_{c_i}} Z_i(v)}{\sum_{v \in \mathcal{G}_{c_j}} Z_j(v)}$. The definition of k employs the fact that the entries of Z_j agree in sign. By Lemma 1, the system is uncontrollable no matter how the leaders are selected from the vertex set $\mathcal{V} \setminus \{\mathcal{V}_{c_i}, \mathcal{V}_{c_j}\}$. Note that by Lemma 8, the Laplacian has the form

$$\mathcal{L} = \begin{bmatrix} \mathcal{F}_1 & 0 & \cdots & 0 & \mathcal{R}_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \mathcal{F}_{\gamma-1} & 0 & \vdots \\ 0 & \cdots & 0 & \mathcal{F}_\gamma & \mathcal{R}_\gamma \\ \mathcal{R}_1^T & \cdots & \cdots & \mathcal{R}_\gamma^T & \mathcal{L}_{\gamma+1} \end{bmatrix}, \tag{4}$$

Then the construction of X_{ij} in [3] implies that the coordinates of X_{ij} corresponding to the leader vertices in $\mathcal{V}_l$ are always $0'$s with respect to different $\mathcal{F}_i$ and $\mathcal{F}_j$ (if there is another pair of $\mathcal{F}_i$ and $\mathcal{F}_j$ with the same property). Since Z_i and Z_j in the Fiedler vector X_{ij} are, respectively, the positive eigenvectors of $\mathcal{F}_i$ and $\mathcal{F}_j$, the corresponding $\mathcal{G}_{c_i}$ and $\mathcal{G}_{c_j}$ are two nonzero components in $\mathcal{G} \setminus \mathcal{G}_l$.

Now, suppose condition (iii) holds. In Lemma 6, we take $\mathcal{W} = \mathcal{V}_l$, and denote the two nonzero components by $\mathcal{G}_1, \mathcal{G}_2$. The conclusions (i) and (ii) then follow from Lemma 6.

Suppose there exists some $\mathcal{G}_{c_\psi}$ which is a Perron branch with the smallest eigenvalue of the corresponding principal submatrix of $\mathcal{L}$ equal to $a(\mathcal{G})$. Then $\tau(\mathcal{F}_\psi) = a(\mathcal{G})$. Repeatedly applying Lemma 5 gives rise to $\tau(\mathcal{F}_1) = \cdots = \tau(\mathcal{F}_\psi) = a(\mathcal{G}), \psi \in \{1, \cdots, \gamma\}$. Since the system matrix is $\mathcal{F} = diag\{\mathcal{F}_1, \cdots \mathcal{F}_\gamma\}$, $a(\mathcal{G})$ is an eigenvalue of $\mathcal{F}$ with multiplicity ψ. By Lemma 7, the multi-agent system is uncontrollable when $\psi \geq 2$. Moreover, since $\tau(\mathcal{F}_1) = \cdots = \tau(\mathcal{F}_\psi) = a(\mathcal{G})$, the previous arguments show that the aforementioned X_{ij} is a Fiedler vector of $\mathcal{L}$, where arbitrary $i, j \in \{1, \cdots, \psi\}, i \neq j$. The system is then uncontrollable with leaders selected from an arbitrary $\mathcal{V} \setminus \{\mathcal{V}_i, \mathcal{V}_j\}$. Note that in this case $\mathcal{V} \setminus \{\mathcal{V}_i, \mathcal{V}_j\}$ varies with respect to different pair of $i, j \in \{1, \cdots, \psi\}$.

Remark 2: It can be seen that condition (ii) is equivalent to the condition involved in the assertion (b). The assertion (b) is stated independently in Theorem 2 to emphasize the fact that the existence of a Perron branch with the smallest eigenvalue of the corresponding principal submatrix of $\mathcal{L}$ equal to $a(\mathcal{G})$

leads to the uncontrollability of the system in most cases, i.e., when $\psi \geq 2$. It can be understood that to satisfy the conditions in the above Theorem 2, the leader vertex set $\mathcal{V}_l$ should contain an articulation.

C. Downer branch and controllability

This subsection devotes to the graph theoretical interpretation of the controllability. In particular, the graphical implication of the algebraic condition given in Lemma 7 will be illustrated. The following notations are required. Denote by $A(\alpha)(A[\alpha])$ the principal submatrix of A resulting from deletion (retention) of the rows and columns α, where $\alpha \subseteq \{1, \cdots, N+l\}$ is an index set of agents. In case α is a singleton $\{i\}$, the $A(\{i\})$ is abbreviated to $A(i)$. We see that $A(v)$ corresponds to the subgraph $\mathcal{T} - v$ of $\mathcal{T}$. Denote by $m_A(\lambda)$ the multiplicity of an eigenvalue λ of A.

Definition 3: We call a branch $\mathcal{T}_0$ of $\mathcal{T}$ at v in the direction of u_0, satisfying the requirement $m_{A[\mathcal{T}_0]}(\lambda) = m_{A[\mathcal{T}_0 \backslash u_0]}(\lambda) + 1$, a downer branch at v for the eigenvalue λ, where u_0 is a neighbor of v in $\mathcal{T}_0$; the vertex u_0 is called a downer vertex.

Theorem 3: For a multi-agent system with tree interconnection graph, the system is uncontrollable if and only if there is a downer branch at some vertex.

Proof The Laplacian can be written as $\mathcal{L} = \begin{bmatrix} \mathcal{F} & \mathcal{R} \\ \mathcal{R}^T & \mathcal{L}_l \end{bmatrix}$. By Lemma 7, the system is uncontrollable if and only if there is an eigenvector x of $\mathcal{F}$, which is (simultaneously) orthogonal to $\mathcal{R}$, i.e., $\mathcal{F}x = \lambda x, x^T\mathcal{R} = 0$. Since $\mathcal{L}[x^T, 0]^T = [x^T\mathcal{F}^T, x^T\mathcal{R}]^T$, we see that

$$\mathcal{F}x = \lambda x,\ x^T\mathcal{R} = 0 \Longleftrightarrow \mathcal{L}[x^T, 0]^T = \lambda[x^T, 0]^T. \quad (5)$$

In the sequel, we are to show the equivalence between the right hand side of (5) and the existence of a downer branch.

Suppose Laplacian $\mathcal{L}$ has an eigenvector $\widetilde{x} \triangleq [x^T, 0]^T$ associated with the eigenvalue λ, where the coordinates of $\widetilde{x}$ corresponding to the leader agents are zeros. Since $\widetilde{x}$ is a nonzero vector and the tree interconnection graph $\mathcal{T}$ is connected, there exists a pair of adjacent vertices k and l with $x_k = 0, x_l \neq 0$. (Note that the index k may be different from the leader agent n.) Let $\mathcal{B}$ be the branch of $\mathcal{T}$ at k that contains vertex l. Then $\mathcal{L}\widetilde{x} = \lambda\widetilde{x}$ implies

$$\mathcal{L}[\mathcal{B}]\widetilde{x}[\mathcal{B}] = \lambda\widetilde{x}[\mathcal{B}], \quad (6)$$

and the entry of $\widetilde{x}$ corresponding to the vertex l is nonzero, where $\mathcal{L}[\mathcal{B}]$ represents the principal submatrix of $\mathcal{L}$ whose rows and columns correspond to the vertices of $\mathcal{B}$ and $\widetilde{x}[\mathcal{B}]$ has the meaning in the same vein. Assume, without loss of generality, that the first entry of $\widetilde{x}[\mathcal{B}]$ is nonzero (Note that the first entry of $\widetilde{x}[\mathcal{B}]$ corresponds to the vertex v_l with $x_l \neq 0$). It follows from (6) that $(\mathcal{L}[\mathcal{B}]-\lambda I)\widetilde{x}[\mathcal{B}] = 0$, and accordingly the 1th column of $\mathcal{L}[\mathcal{B}] - \lambda I$ is a linear combination of all the other columns of $\mathcal{L}[\mathcal{B}] - \lambda I$. As a consequence,

$$\dim(CS(\mathcal{L}[\mathcal{B}] - \lambda I)) = \dim(CS((\mathcal{L}[\mathcal{B}] - \lambda I)[:, \overline{\{1\}}])), \quad (7)$$

where $CS(\mathcal{L}[\mathcal{B}]-\lambda I)$ denotes the column space of $\mathcal{L}[\mathcal{B}]-\lambda I$; $\overline{\{1\}}$ represents the index set $\{1, \cdots, n\}\backslash\{1\}$, and $(\mathcal{L}[\mathcal{B}] - \lambda I)[:, \overline{\{1\}}]$ is the submatrix of $\mathcal{L}[\mathcal{B}] - \lambda I$ consisting of columns indexed by $\overline{\{1\}}$. Let $\nu(A)$ represent the nullity of matrix A. We have from (7) that

$$\begin{aligned} &\nu((\mathcal{L}[\mathcal{B}] - \lambda I)[:, \overline{\{1\}}]) \\ =&(|\mathcal{B}| - 1) - rank((\mathcal{L}[\mathcal{B}] - \lambda I)[:, \overline{\{1\}}]) \\ =&\nu((\mathcal{L}[\mathcal{B}] - \lambda I)) - 1, \end{aligned} \quad (8)$$

where $|\mathcal{B}|$ is the cardinality of branch $\mathcal{B}$. Since $\mathcal{L}[\mathcal{B}] - \lambda I$ is symmetric and the 1th column of $\mathcal{L}[\mathcal{B}] - \lambda I$ is a linear combination of all the other columns of $\mathcal{L}[\mathcal{B}] - \lambda I$, the 1th row of $(\mathcal{L}[\mathcal{B}] - \lambda I)[:, \overline{\{1\}}]$ is a linear combination of $(\mathcal{L}[\mathcal{B}] - \lambda I)(1)$. Accordingly, $\dim(RS((\mathcal{L}[\mathcal{B}] - \lambda I)(1))) = \dim(RS((\mathcal{L}[\mathcal{B}]-\lambda I)[:, \overline{\{1\}}]))$, where $RS(\cdot)$ denotes the row space of a matrix. Hence

$$\nu((\mathcal{L}[\mathcal{B}] - \lambda I)(1)) = \nu((\mathcal{L}[\mathcal{B}] - \lambda I)[:, \overline{\{1\}}]). \quad (9)$$

Moreover, by (8) and (9),

$$\nu((\mathcal{L}[\mathcal{B}] - \lambda I)(1)) = \nu((\mathcal{L}[\mathcal{B}] - \lambda I)) - 1. \quad (10)$$

That is, $m_{\mathcal{L}[\mathcal{B}](1)}(\lambda) = m_{\mathcal{L}[\mathcal{B}]}(\lambda) - 1$. Hence, vertex v_1 is a downer vertex of $\mathcal{L}[\mathcal{B}]$, and accordingly, $\mathcal{B}$ is a downer branch.

The above arguments show that $\mathcal{L}[x^T, 0]^T = \lambda[x^T, 0]^T$ implies the existence of a downer branch at vertex v_k in the direction of v_l, where the corresponding entries satisfy $x_k = 0, x_l \neq 0$.

For the contrary, assume $\mathcal{B}$ is a downer branch at vertex v. Below, we shall first show $e_v \notin CS(\mathcal{L}[\mathcal{B}] - \lambda I)$, where e_v is the identity vector associated with vertex v. This will be proved by contradiction. Suppose $e_v \in CS(\mathcal{L}[\mathcal{B}] - \lambda I)$. Since $\mathcal{L}[\mathcal{B}] - \lambda I$ is symmetric, $e_v^T \in RS(\mathcal{L}[\mathcal{B}] - \lambda I)$, and each vector in $RS(\mathcal{L}[\mathcal{B}] - \lambda I)$ is orthogonal to the vectors in $NS(\mathcal{L}[\mathcal{B}] - \lambda I)$. So, each vector in $NS(\mathcal{L}[\mathcal{B}] - \lambda I)$ is orthogonal to e_v, and accordingly, $NS(\mathcal{L}[\mathcal{B}]-\lambda I) \subseteq e_v^{\perp}$. As a consequence, the coordinate corresponding to the vertex v in each vector in $NS(\mathcal{L}[\mathcal{B}] - \lambda I)$ is equal to zero. Let $\xi \in NS(\mathcal{L}[\mathcal{B}] - \lambda I)$. One has $\mathcal{L}[\mathcal{B}]\xi = \lambda\xi$. Since the entry corresponding to the vertex v in ξ is zero, it follows that $\mathcal{L}[\mathcal{B}](v)\xi(v) = \lambda\xi(v)$. Hence, each eigenvalue λ of $\mathcal{L}[\mathcal{B}]$ is also an eigenvalue of $\mathcal{L}[\mathcal{B}](v)$. Consequently, $m_{\mathcal{L}[\mathcal{B}](v)}(\lambda) \geq m_{\mathcal{L}[\mathcal{B}]}(\lambda)$, a contradiction to the fact that v is a downer vertex of $\mathcal{B}$, i.e., $m_{\mathcal{L}[\mathcal{B}](v)}(\lambda) = m_{\mathcal{L}[\mathcal{B}]}(\lambda) - 1$.

Since $e_v \notin CS(\mathcal{L}[\mathcal{B}] - \lambda I)$ and v is labeled as the first vertex in $\mathcal{B}$, the first column of $(\mathcal{L} - \lambda I)[\overline{\{1\}}, :]$ is not in the span of the remaining columns of $(\mathcal{L} - \lambda I)[\overline{\{1\}}, :]$, where $(\mathcal{L} - \lambda I)[\overline{\{1\}}, :]$ denotes the submatrix of $\mathcal{L} - \lambda I$ consisting of rows indexed by $\overline{\{1\}}$. Assume that $\mathcal{L} - \lambda I$ has a block decomposition form as follows:

$$\mathcal{L} - \lambda I = \begin{bmatrix} l_{11} - \lambda & h^T \\ h & \mathcal{L}(1) - \lambda I \end{bmatrix}.$$

Since h is not a linear combination of $\mathcal{L}(1)-\lambda I$, sequentially extending $\mathcal{L}(1) - \lambda I$ by the column h and then by the row $[l_{11} - \lambda, h^T]$ increases the rank each time. Thus, $rank(\mathcal{L} - \lambda I) = rank(\mathcal{L}(1)-\lambda I)+2$, which leads to $\nu((\mathcal{L}-\lambda I)(1)) = \nu((\mathcal{L} - \lambda I)) + 1$.

Next, we are to show that the first column of $\mathcal{L} - \lambda I$ is not in $CS((\mathcal{L}-\lambda I)[:, \overline{\{1\}}])$. Suppose to the contrary that the first column of $\mathcal{L}-\lambda I$ is a linear combination of $((\mathcal{L}-\lambda I)[:, \overline{\{1\}}])$. Then $\dim(CS(\mathcal{L} - \lambda I)) = \dim(CS((\mathcal{L} - \lambda I)[:, \overline{\{1\}}]))$. Consequently, following the same reasonings for (10) gives rise to $\nu((\mathcal{L} - \lambda I)(1)) = \nu((\mathcal{L} - \lambda I)) - 1$, a contradiction.

Now, it has been shown that the first column of $\mathcal{L} - \lambda I$ is not in $CS((\mathcal{L} - \lambda I)[:, \overline{\{1\}}])$. If $\eta \in NS(\mathcal{L} - \lambda I)$ whose first coordinate is nonzero, then the first column of $\mathcal{L} - \lambda I$ is in $CS((\mathcal{L} - \lambda I)[:, \overline{\{1\}}])$. This is a contradiction. So, there is an $\eta \in NS(\mathcal{L} - \lambda I)$ with its first entry being zero. Therefore, η is an eigenvector of $\mathcal{L}$, and then the system is uncontrollable if the first agent takes the leader role.

The above analysis tells us that the existence of a downer branch implies that there is an eigenvector of the Laplacian $\mathcal{L}$, which has at least one zero coordinate. Then the system is uncontrollable with the agents corresponding to the zero coordinates taking the leaders role.

Remark 3: Although for the single leader case, Theorem 3 can also be derived by combining Lemma 1 with Corollary 3.3 of [14] and Theorem 8 of [13], the graph theoretical implication of the algebraic condition presented in Lemma 7 (or Theorem IV.1 of [23]) cannot be fully revealed if the proof is conducted in this way. To get a direct and intuitive graphical feeling for the algebraic condition and inspired by the results in [13], [14], we present here a complete and detailed proof for the equivalence between the existence of a downer branch and the above algebraic condition. In addition, the proof reveals the relationship between uncontrollable leader vertices and the vertex at which the downer branch occurs.

IV. Conclusions

This paper studies interconnection topology structures for coordination and control of multi-agent networks. Graph theoretical characterizations are given for the controllability of multi-agent networks in terms of downer and Perron branches of an associated interconnection graph. In particular, a graph theoretical interpretation is presented for the algebraic condition of controllability with respect to tree topology. The robustness of multi-agent controllability is also studied when edge failures occur in the graph.

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