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Multi-agent Controllability with Tree Topology

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Abstract— **This paper focuses on the controllability of multi-agent systems with tree topology. The connection on controllability/uncontrollability between a single leader and multiple leaders is revealed, as well as the relationship between controllability and the common eigenvalues of components is analyzed. After that, a constructive approach is outlined to find out various uncontrollable subgraphs when the whole interconnection tree graph is uncontrollable, and a necessary and sufficient condition is presented for multi-agent controllability with tree topology.**

I. INTRODUCTION

The distributed coordination and control of multi-agent networked systems has recently received considerable attention, see e.g., [1], [2], [3], [4], [5]. Since communications between agents are essential for the coordination and co-operation among agents, the characterization of properties of multi-agent systems relies heavily on the interconnection topology structure of the graph associated with the network. This motives the study of controllable/uncontrollable interconnection topologies for multi-agent systems in the paper.

Controllability is a crucial concept in classical control. Multi-agent controllability problem was first proposed by Tanner in [6], where necessary and sufficient algebraic conditions were derived. He also pointed out that the building of controllable interconnection topologies calls for a graph theoretic characterization of the controllability property. From then on, the controllability study of multi-agent networks aroused more and more attention. In [7], [8], [9], [10], the relationship between symmetric structure of the network and the controllability of the corresponding multi-agent systems was explored, as well as algebraic conditions on controllability in terms of, e.g., eigenvalues. Controllability under switching topology and time-delay was studied in [11], [12], [13], and uncontrollable topology structures and graph theoretic properties were given in [14], [15]. Recently, a controllability decomposition through quotient graphs was presented by Martini, Egerstedt and Bicchi [16]. The controllability and observability of several interconnection configurations were analyzed in [17]. The structural controllability and higher order integrator agents controllability were studied in [18] and [19], respectively. The controllability of state-dependent dynamic graphs and the observability of networks were investigated in [20][21].

Inspired by these works, we study multi-agent controllability problem for tree interconnection topology. The problem is analyzed under leader-follower framework. Necessary and sufficient conditions are derived from the viewpoint of both leaders role and the common eigenvalues of components. To further examine the structural properties of tree topology with respect to controllability, we present a partition of tree graph by taking advantage of the eigenvector property of trees. It turns out that uncontrollable subgraphs could be found out via several designed steps. Finally, a necessary and sufficient condition is derived for multi-agent controllability with tree interconnection graph.

II. PRELIMINARIES

A. Graph preliminaries

We denote by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ an undirected graph, with $\mathcal{V} = \{v_1, \cdots, v_n\}$ being the node set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the edge set, where an edge is an unordered pair of distinct nodes of $\mathcal{V}$. Two nodes v_i and v_j are neighbors if $(v_i, v_j) \in \mathcal{E}$, and the neighboring relation is indicated with $v_j \sim v_i$. In this case we say that v_j is incident to v_i. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ be two graphs. We call $\mathcal{G}'$ a subgraph of $\mathcal{G}$ if $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$, and we denote this by $\mathcal{G}' \subseteq \mathcal{G}$. A subgraph $\mathcal{G}'$ is said to be induced from the original graph $\mathcal{G}$ if it is obtained by deleting a subset of nodes and all the edges connecting to those nodes. $\mathcal{G}' \subseteq \mathcal{G}$ is a spanning subgraph of $\mathcal{G}$ if $\mathcal{V}' = \mathcal{V}$. An undirected graph is said to be connected if there exists a path between any two distinct nodes of the graph. An induced subgraph of an undirected graph, which is maximal and connected, is said to be a connected component of the undirected graph. A tree is a connected graph which contains no circuits.

In association with $\mathcal{G}$, a Laplacian matrix $\mathcal{L} = (l_{ij}) \in \mathbf{R}^{n \times n}$ is defined by

$$l_{ij} = \begin{cases} -a_{ij}, & \text{if } i \neq j \\ \sum_{j=1, j\neq i}^{n} a_{ij}, & \text{if } i = j \end{cases} \tag{1}$$

where $a_{ij} > 0$ is a weight if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$ if $(j, i) \notin \mathcal{E}$. A symmetric real $n \times n$ matrix A is said to be a *generalized* Laplacian of an undirected graph $\mathcal{G}$ if $a_{ij} < 0$ when v_i and v_j are adjacent vertices of $\mathcal{G}$ and $a_{ij} = 0$ when i and j are distinct and not adjacent. There are no constraints on the diagonal entries of A. Clearly, the Laplacian defined in (1) is a special kind of generalized Laplacian matrix. Throughout the paper all graphs are assumed to be simple, i.e., graph without multiple or directed edges, and without loops, and

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all matrices are assumed to be real. The eigenvalues and eigenvectors of $\mathcal{G}$ are, respectively, referred to those of Laplacian $\mathcal{L}$.

B. Problem formulation

Consider a multi-agent system given by

$$\begin{cases} \dot{x}_i = u_i, & i = 1, \dots, N \\ \dot{x}_{N+j} = u_{N+j}, & j = 1, \dots, l \end{cases} \tag{2}$$

where x_i is the state of the ith agent; N and l are the number of followers and leaders, respectively, $i = 1, \cdots, N + l$.

Definition 1: [6] The interconnection graph, $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, is an undirected graph consisting of a set of nodes, $\mathcal{V} = \{v_1, \dots, v_N, v_{N+1}, \dots, v_{N+l}\}$, indexed by the agents in the group; and a set of edges, $\mathcal{E} = \{(v_i, v_j) \in \mathcal{V} \times \mathcal{V} | v_i \sim v_j\}$, containing unordered pairs of nodes that correspond to interconnected agents.

The topology of an interconnection graph $\mathcal{G}$ is said to be fixed if each node of $\mathcal{G}$ has a fixed neighbor set. Let $\mathcal{N}_i = \{j \mid v_i \sim v_j; j \neq i\}$, which is the neighboring set of v_i; and define the protocol as follows:

$$u_i = -\sum_{j \in \mathcal{N}_i} a_{ij}(x_i - x_j). \tag{3}$$

Take $x_{N+1}, \cdots, x_{N+l}$ to play leaders role, and rename the agents as

$$\begin{cases} y_i \triangleq x_i, & i = 1, \dots, N; \\ z_j \triangleq x_{N+j}, & j = 1, \dots, l. \end{cases}$$

Let y, z and u denote the stack vectors of all y_i, z_j, and u_{N+j}, respectively, $i = 1, \cdots, N$; $j = 1, \dots, l$. In this leader-follower framework, the leaders' neighbors still obey (3), but the leaders are free of such a constraint and are allowed to pick u_{N+j} arbitrarily, $j = 1, \dots, l$. Then, under protocol (3), the multi-agent system (2) reads

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = -\begin{bmatrix} \mathcal{F} & \mathcal{R} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix},$$

where $\mathcal{F}$ is the matrix obtained from the Laplacian matrix $\mathcal{L}$ of $\mathcal{G}$ after deleting the last l rows and l columns. $\mathcal{R}$ is the $N \times l$ submatrix consisting of the first N elements of the deleted columns. The dynamics of the followers that correspond to the y component of the equation is extracted as

$$\dot{y} = -\mathcal{F}y - \mathcal{R}z. \tag{4}$$

Definition 2: The multi-agent system (2) is said to be controllable under leaders $x_{N+j}, j = 1, \dots, l$, and fixed topology if system (4) is controllable under control input z. Otherwise, it is said to be uncontrollable.

III. MAIN RESULTS

Lemma 1: (Lemma 2.2, [8]) Suppose the interconnection graph $\mathcal{G}$ is connected, the multi-agent system (2) is controllable if and only if $\mathcal{L}$ and $\mathcal{F}$ do not share any common eigenvalues.

Lemma 2: [14] The multi-agent system with (undirected) weighted interconnection graphs is controllable if and only if there is no eigenvector of Laplacian matrix $\mathcal{L}$ taking 0 on the elements corresponding to the leaders.

The following notations are borrowed from [22]. If $\alpha \subseteq \{1, \cdots, N + l\}$ is an index set of agents, we denote the principal submatrix of A resulting from deletion (retention) of the rows and columns α by $A(\alpha)(A[\alpha])$. If α consists of a single index i, the $A(\{i\})$ is abbreviated to $A(i)$. We see that $A(v)$ corresponds to the subgraph $\mathcal{T} - v$ of $\mathcal{T}$. Here, $\mathcal{T}$ represents a tree graph. Let $m_A(\lambda)$ denotes the multiplicity of an eigenvalue λ of A. If $m_{A(i)}(\lambda) = m_A(\lambda)+1$, the vertex i in $\mathcal{T}$ is said to be a Parter vertex for an eigenvalue λ and a Hermitian matrix A whose graph is $\mathcal{T}$. A collection $\alpha \subseteq N$ is said to be a Parter set when $m_{A(\alpha)}(\lambda) = m_A(\lambda) + |\alpha|$.

Lemma 3: (Theorem 14, [22]) Let A be a Hermitian matrix whose graph is a tree $\mathcal{T}$ and let λ be an eigenvalue of A. Then, there is a vertex v of $\mathcal{T}$ such that A and $A(v)$ share a common eigenvalue λ if and only if there is a Parter set S of cardinality $k \geq 1$ such that λ is an eigenvalue of $m_A(\lambda) + k$ direct summands of $A(S)$.

Proposition 1: For a multi-agent system with tree interconnection graph $\mathcal{T}$, the following conclusions hold:

(i) the system is uncontrollable under a single leader if and only if there exist a group of vertices $v_1, \cdots, v_k$, such that the system is uncontrollable if $v_1, \cdots, v_k$ are taken to play the leaders role.

(ii) the system is uncontrollable if and only if there exist a group of vertices $v_1, \cdots, v_k$, and an eigenvalue λ of $\mathcal{L}$, so that $m_{\mathcal{L}}(\lambda) + k$ components of $\mathcal{T} - \{v_1, \cdots, v_k\}$ share λ as a common eigenvalue.

Proof: Part I: The Laplacian $\mathcal{L}$ of a tree $\mathcal{T}$ is a Hermitian matrix whose graph is a tree. By Lemma 1, the multi-agent system is uncontrollable under the single leader v if and only if $\mathcal{L}$ and $\mathcal{L}(v)$ share a common eigenvalue λ. By Lemma 3, $\mathcal{L}$ and $\mathcal{L}(v)$ share a common eigenvalue λ if and only if there is a Parter set S of cardinality $k \geq 1$ such that λ is an eigenvalue of $m_{\mathcal{L}}(\lambda) + k$ direct summands of $\mathcal{L}(S)$. Since each direct summand of $\mathcal{L}(S)$ corresponds to a component of $\mathcal{T} - S$, λ is a common eigenvalue of $m_{\mathcal{L}}(\lambda)+k$ components of $\mathcal{T}-S$. Accordingly, λ is a multiple eigenvalue of $\mathcal{L}(S)$ with multiplicity not less than $m_{\mathcal{L}}(\lambda)+k$. Suppose the Parter set S consists of vertices $v_1, \cdots, v_k$. It follows that $\mathcal{L}$ and $\mathcal{L}(v_1, \cdots, v_k)$ share a common eigenvalue λ. Again, by Lemma 1, the system is uncontrollable under leaders $v_1, \cdots, v_k$.

For the converse, suppose the system is uncontrollable under leaders $v_1, \cdots, v_k$. By Lemma 2, there is an eigenvector y associated with an eigenvalue λ of $\mathcal{L}$, and the eigenvector can be written in the form of $y = [y_1, \cdots, y_{N+l-k}, \underbrace{0, \cdots, 0}_{k}]^T$. It follows from $\mathcal{L}y = \lambda y$ and the specific form of y that λ is also an eigenvalue of $\mathcal{L}(v_i)$ for any $i \in \{1, \cdots, k\}$. Then $\mathcal{L}$ and $\mathcal{L}(v_i)$ share a common eigenvalue λ. By Lemma 1, the system is uncontrollable under any single leader v_i, $i = 1, \cdots, k$. The assertion then follows from the above arguments.

Part II: In view of the above arguments, if the system is uncontrollable, take an arbitrary leader agent v, the system is still uncontrollable under the single leader v. By Lemma 1, $\mathcal{L}$ and $\mathcal{L}(v)$ share a common eigenvalue. The conclusion then follows by combining Lemmas 3, 1 with the arguments for the proof of the first part. ■

Leader-follower induced partition: Let $\mathcal{G}_{c_1}, \ldots, \mathcal{G}_{c_\gamma}$ be the γ connected components of the follower subgraph $\mathcal{G}_f$, with $\mathcal{G}_{c_i}$ on the node set $\{v_{n_{i-1}+1}, \ldots, v_{n_i}\}$, $i = 1, \ldots, \gamma$; $n_0 = 0, n_\gamma = N$; and $\mathcal{G}_l$ on the node set $\mathcal{V}_l = \{v_{N+1}, \ldots, v_{N+l}\}$. Denote by $\mathcal{G}(i)$ an induced subgraph of $\mathcal{G}$, which is on the node set $\{v_{n_{i-1}+1}, \ldots, v_{n_i}, v_{N+1}, \ldots, v_{N+l}\}$. That is, the node set of $\mathcal{G}(i)$ is the union of those of $\mathcal{G}_{c_i}$ and $\mathcal{G}_l$. Then, $\mathcal{G}(1), \cdots, \mathcal{G}(\gamma)$ are considered to constitute a 'partition' of $\mathcal{G}$ in the sense that $\mathcal{G}$ is partitioned into γ induced subgraphs $\mathcal{G}(1), \cdots, \mathcal{G}(\gamma)$, with each one having the same leader subgraph $\mathcal{G}_l$ and the union of them coincides with $\mathcal{G}$.

With this 'partition' of interconnection graph $\mathcal{G}$, the Laplacian matrix can be written in the form of

$$\mathcal{L} = \begin{bmatrix} \mathcal{F}_1 & 0 & \cdots & \cdots & 0 & \mathcal{R}_1 \\ 0 & \mathcal{F}_2 & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \mathcal{F}_{\gamma-1} & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & \mathcal{F}_\gamma & \mathcal{R}_\gamma \\ \mathcal{R}_1^T & \cdots & \cdots & \cdots & \mathcal{R}_\gamma^T & \mathcal{L}_{\gamma+1} \end{bmatrix}, \quad (5)$$

where $\mathcal{F}_i$ and the matrix pair $(\mathcal{F}_i, \mathcal{L}_{\gamma+1})$ correspond to $\mathcal{G}_{c_i}$, and $\mathcal{G}(i)$, respectively.

Lemma 4: [15] If multi-agent system (2) with fixed topology is controllable, then the interconnection graph is leader-follower connected, and each subgraph $\mathcal{G}(i)$ is controllable, where $i \in \{1, \ldots, \gamma\}$; γ is the number of connected components in $\mathcal{G}_f$.

For a tree interconnection graph $\mathcal{T}$, we introduce the following partition.

Vanishing coordinates induced partition: Let Z be the set of all vanishing coordinates of an eigenvector. Take subgraph on Z to be the leader subgraph and $\mathcal{T} - Z$ to be the follower subgraph. Let $\mathcal{T}_{c_1}, \cdots, \mathcal{T}_{c_\delta}$ be the components of $\mathcal{T} - Z$. Then subgraphs $\mathcal{T}(1), \cdots, \mathcal{T}(\delta)$ can be defined in the same way as $\mathcal{G}(1), \cdots, \mathcal{G}(\gamma)$ in the leader-follower related partition. We see that $\mathcal{T}(1), \cdots, \mathcal{T}(\delta)$ can be considered to constitute another 'partition' for the tree interconnection graph $\mathcal{T}$.

Let v be an arbitrary vertex in Z. Then either v is incident to a vertex in some $\mathcal{T}_{c_i}, i \in \{1, \cdots, \delta\}$; or v is not incident to any vertex of any $\mathcal{T}_{c_i}$. Accordingly, vertices in Z can be divided into two categories. Denote by Z_1 the set of vertices of the first category, and Z_2 the set of second category, i.e., vertices in Z which are not incident to any vertex in any $\mathcal{T}_{c_i}$.

A vertex v is said to be incident to a subgraph if it is incident to a vertex in the very subgraph. In this case, we will also say that the subgraph is incident to the vertex v. Denote by $Z_{1,i}$ the set of vertices which are incident to the component $\mathcal{T}_{c_i}$. Each $Z_{1,i}$ is not empty since tree graph is connected. Let $\mathcal{T}'_{c_i}$ be the subgraph which is on the vertices of $\mathcal{T}_{c_i}$ and $Z_{1,i}$, $i = 1, \cdots, \delta$.

Next, we introduce another subgraph with respect to each fixed vertex $v_j \in Z_1$. Let $\mathcal{T}^{[j]}$ represent the subgraph which consists of v_j itself and all the incident components $\mathcal{T}_{c_j}$ of v_j, $j \in \{1, \cdots, \delta\}$. By taking advantage of the aforementioned two kinds of 'partitions', these two kinds of subgraphs $\mathcal{T}^{[j]}$ and $\mathcal{T}'_{c_i}$ are to be employed to characterize the 'smaller' uncontrollable subgraphs once the original interconnection graph is uncontrollable. The following result contributes to the understanding of structural property of uncontrollable graphs.

Theorem 1: For a multi-agent system with tree interconnection graph $\mathcal{T}$, the following assertions hold:

(i) if the system is uncontrollable, then for each vertex $v_j \in Z_1$, there is a group of vertices $\omega_{j,1}, \cdots, \omega_{j,k}$ of $\mathcal{T}^{[j]}$ such that $\mathcal{T}^{[j]}$ is uncontrollable if $\omega_{j,1}, \cdots, \omega_{j,k}$ are taken to play leaders role.

(ii) if $Z_{1,i} \cap Z_{1,j} = \emptyset$ for some fixed i, where $i \neq j$, $\forall j \in \{1, \cdots, \delta\}$, $\emptyset$ is the empty set; then the system is uncontrollable if and only if the corresponding $\mathcal{T}'_{c_i}$ is uncontrollable with leaders selected from $Z_{1,i}$.

Proof: Part I: In consideration of clear expression, the proof is implemented by taking the following five steps.

Step 1: By Lemma 2, if the system is uncontrollable under l multiple leaders, there is an eigenvector y associated with an eigenvalue λ of $\mathcal{L}$, which is, without loss of generality, in the form of $y = [y_1, \cdots, y_N, \underbrace{0, \cdots, 0}_{l}]^T$. Since Z is the set of all vanishing coordinates of the eigenvector, y can be further written as

$$y = [y_1, \cdots, y_\tau, \underbrace{0, \cdots, 0}_{|Z|}], \quad (6)$$

where $y_i \neq 0$, $i = 1, \cdots, \tau$; $\tau \triangleq N + l - |Z|$. Let $\mathcal{T}_{c_1}, \cdots, \mathcal{T}_{c_\delta}$ be the components of $\mathcal{T} - Z$. The Laplacian can be conformably written as

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_1 & 0 & \cdots & 0 & \mathcal{R}_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \mathcal{L}_{\delta-1} & 0 & \vdots \\ 0 & \cdots & 0 & \mathcal{L}_\delta & \mathcal{R}_\delta \\ \mathcal{R}_1^T & \cdots & \cdots & \mathcal{R}_\delta^T & \mathcal{L}_{\delta+1} \end{bmatrix}, \quad (7)$$

and the eigenvector can be conformably partitioned as $y = [\eta_1^T, \cdots, \eta_\delta^T, 0^T]^T$, where η_i are vectors corresponding to $\mathcal{L}_i$, $i = 1, \cdots, \delta$; and the vector 0 corresponds to $\mathcal{L}_{\delta+1}$. Then $\mathcal{R}_i$ is $(m_i - m_{i-1}) \times |Z|$, where $m_i - m_{i-1}$ is the number of vertices of $\mathcal{T}_{c_i}$. To proceed with the proof, we need to write out $\mathcal{R}_i$, which is as follows:

$$\mathcal{R}_i = \begin{bmatrix} r_{m_{i-1}+1,\tau+1} & \cdots & r_{m_{i-1}+1,N+l} \\ \vdots & \ddots & \vdots \\ r_{m_i,\tau+1} & \cdots & r_{m_i,N+l} \end{bmatrix}, \ i = 1, \cdots, \delta, \quad (8)$$

where $m_0 = 0, m_\delta = N + l - |Z|$. One advantage of this vanishing coordinates related partition is the property $y_i \neq 0$ for any $i \in \{1, \cdots, \tau\}$, which is important for subsequent derivation.

Step 2: For the leader agents in Z, they can be divided into two categories: one is the vertex which is incident to another vertex in some component $\mathcal{T}_{c_i},\ i \in \{1, \cdots, \delta\}$; the other is the vertex which does not belong to the first category.

Let us first consider the leader agents belonging to the second category. Since the vertices of the second category are not incident to vertices in any component $\mathcal{T}_{c_i}$, the row vectors corresponding to these vertices in $[\mathcal{R}_1^T, \cdots, \mathcal{R}_\delta^T]$ are all zero vectors. For the convenience of statement, we assume that the vertex set of the second category consists of the last two vertices v_{N+l-1}, v_{N+l}. The general case can be proved in the same manner. Then the remaining $|Z| - 2$ vertices $v_{\tau+1}, \cdots, v_{N+l-2}$ constitute the vertex set of the first category.

Consider $\mathcal{L}(v_{N+l-1}, v_{N+l})$, which corresponds to the subgraph $\mathcal{T} - \{v_{N+l-1}, v_{N+l}\}$. The interconnection relationship (information flow) between components $\mathcal{T}_{c_i}$ and the leader agents group remains unchanged from $\mathcal{T}$ to $\mathcal{T} - \{v_{N+l-1}, v_{N+l}\}$. This is because v_{N+l-1}, v_{N+l} are vertices of second category which are not incident to vertices of any component $\mathcal{T}_{c_i}$, and accordingly the $(N+l-1)$th and $(N+l)$th row vectors of $[\mathcal{R}_1^T, \cdots, \mathcal{R}_\delta^T]$ are both zeroes. This, together with $\mathcal{L}y = \lambda y$ and the specific structure of y, yields that $\mathcal{L}$, $\mathcal{L}(v_{N+l})$, and $\mathcal{L}(v_{N+l-1}, v_{N+l})$ share the common eigenvalue λ.

We see that the removing of vertices of second category does not affect the components $\mathcal{T}_{c_1}, \cdots, \mathcal{T}_{c_\delta}$ of $\mathcal{T} - Z$, as well as the controllability of the original interconnection graph $\mathcal{T}$. This is because with respect to the second category vertices, the corresponding columns associated with these vertices in $[\mathcal{R}_1^T, \cdots, \mathcal{R}_\delta^T]^T$ are all zero vectors. These zero columns do not affect the rank of the controllability matrix.

Step 3: We proceed to consider $\mathcal{L}(v_{N+l-2}, v_{N+l-1}, v_{N+l})$, the information flow between components $\mathcal{T}_{c_i}$ and the leader agents in $\mathcal{T} - \{v_{N+l-2}, v_{N+l-1}, v_{N+l}\}$ is altered, which is different from that between $\mathcal{T}_{c_i}$ and the leader agents group in $\mathcal{T} - \{v_{N+l-1}, v_{N+l}\}$. More specifically, the removing of vertex v_{N+l-2} together with its incident edges in $\mathcal{T} - \{v_{N+l-1}, v_{N+l}\}$ may bring the following changes for components:

(i) 'isolated' components among $\mathcal{T}_{c_i}$ occur, $i = 1, \cdots, \delta$. In this case, only leader agent v_{N+l-2} is incident to a vertex in each of these isolated components in the original subgraph $\mathcal{T} - \{v_{N+l-1}, v_{N+l}\}$. So isolated components occur once v_{N+l-2} is removed from $\mathcal{T} - \{v_{N+l-1}, v_{N+l}\}$. Note that these isolated components only consist of follower vertices.

(ii) 'separated' subtrees occur. Each of these subtrees contains both follower and leader agent vertices. This is what separated subtree differs from the aforementioned isolated component. The latter only consists of follower vertices. In this case, at least two leader vertices are incident to at least one of vertices in each separated subtree in the original subgraph $\mathcal{T} - \{v_{N+l-1}, v_{N+l}\}$. So the removing of one leader vertex is not enough to isolate the corresponding component from the leader agents group.

Note that 'isolated' components and 'separated' subtrees may occur simultaneously.

Step 4: Repeat this process, suppose vertices $v_{\tau+2}, \cdots, v_{N+l-2}$ are all successively removed from $\mathcal{T} - \{v_{N+l-1}, v_{N+l}\}$. We consider $\mathcal{T} - \{v_{\tau+2}, \cdots, v_{N+l-2}, v_{N+l-1}, v_{N+l}\}$. At this moment, the remaining $v_{\tau+1}$ is the unique leader, and accordingly only 'isolated' components could occur in $\mathcal{T} - \{v_{\tau+2}, \cdots, v_{N+l}\}$. To simplify presentation, we assume $\delta = 4$, and that the components corresponding to $\mathcal{L}_1$ and $\mathcal{L}_4$ are isolated. The general analysis can be conducted in the same manner. In this case, $\mathcal{L}_2, \mathcal{L}_3$ and the leader vertex $v_{\tau+1}$ constitute a connected subtree. The principle submatrix of Laplacian $\mathcal{L}$, which corresponds to $\mathcal{T} - \{v_{\tau+2}, \cdots, v_{N+l}\}$, is

$$\mathcal{L}(v_{\tau+2}, \cdots, v_{N+l}) = \begin{bmatrix} \mathcal{L}_1 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{L}_2 & 0 & 0 & r_2 \\ 0 & 0 & \mathcal{L}_3 & 0 & r_3 \\ 0 & 0 & 0 & \mathcal{L}_4 & 0 \\ 0^T & r_2^T & r_3^T & 0^T & l_{1,1} \end{bmatrix}. \quad (9)$$

Since $\mathcal{T} - \{v_{\tau+2}, \cdots, v_{N+l}\}$ is a subtree and $\mathcal{T}_2, \mathcal{T}_3$ are components, r_2, r_3 are vectors with a unique element being nonzero, respectively. Partition the eigenvector y conformably with the Laplacian $\mathcal{L}$ in (7) as follows: $y = [\hat{y}_1^T, \cdots, \hat{y}_\delta^T, 0_{\delta+1}^T]^T$. Again, it follows from $\mathcal{L}y = \lambda y$ and the specific form of y that

$$\begin{aligned} &\mathcal{L}(v_{\tau+2}, \cdots, v_{N+l})y(v_{\tau+2}, \cdots, v_{N+l}) \\ =&\lambda y(v_{\tau+2}, \cdots, v_{N+l}), \end{aligned} \quad (10)$$

where $y(v_{\tau+2}, \cdots, v_{N+l}) = [\hat{y}_1^T, \hat{y}_2^T, \hat{y}_3^T, \hat{y}_4^T, 0]^T$. Combining (9) with (10) gives rise to

$$\mathcal{L}^{[\tau+1]} \begin{bmatrix} \hat{y}_2 \\ \hat{y}_3 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} \hat{y}_2 \\ \hat{y}_3 \\ 0 \end{bmatrix}, \quad (11)$$

where $\mathcal{L}^{[\tau+1]} \triangleq \begin{bmatrix} \mathcal{L}_2 & 0 & r_2 \\ 0 & \mathcal{L}_3 & r_3 \\ r_2^T & r_3^T & l_{1,1} \end{bmatrix}$. Since $\mathcal{L}_i$ corresponds to $\mathcal{T}_{c_i}$, $\mathcal{L}^{[\tau+1]}$ is a generalized Laplacian of the induced subgraph on the vertex set union of $v_{\tau+1}$, $\mathcal{T}_{c_2}$, and $\mathcal{T}_{c_3}$. In other words, $\mathcal{L}^{[\tau+1]}$ is a generalized Laplacian of the induced subgraph which consists of the incident components $\mathcal{T}_{c_2}, \mathcal{T}_{c_3}$ of $v_{\tau+1}$ and the vertex $v_{\tau+1}$ itself.

It follows from (11) that $\mathcal{L}^{[\tau+1]}$ and $\begin{bmatrix} \mathcal{L}_2 & 0 \\ 0 & \mathcal{L}_3 \end{bmatrix}$ share the common eigenvalue λ since $\hat{y}_2, \hat{y}_3$ are both nonzero vectors. By Lemma 3, there are k vertices, say $\omega_1, \cdots, \omega_k$ in $\mathcal{T}^{[\tau+1]}$, so that λ is a common eigenvalue of $m_{\mathcal{L}^{[\tau+1]}}(\lambda) + k$ components of $\mathcal{T}^{[\tau+1]} - \{\omega_1, \cdots, \omega_k\}$. As a consequence, if $\omega_1, \cdots, \omega_k$ are chosen to play leaders role in the subtree $\mathcal{T}^{[\tau+1]}$, the corresponding Laplacian and the associated system matrix share the common eigenvalue λ. Moreover, by Lemma 1, $\mathcal{T}^{[\tau+1]}$ is uncontrollable with $\omega_1, \cdots, \omega_k$ being leaders.

Step 5: As mentioned above, the vertex set of the first category is $\{v_{\tau+1}, \cdots, v_{N+l-2}\}$. With respect to each vertex

v_j in this set, there is a subtree $\mathcal{T}^{[j]}$, $j=\tau+1,\cdots,N+l-2$; which is obtained by removing, except v_j, the other leader vertices and their incident edges, as well as the isolated components caused by the removing of vertices in $\mathcal{T}$. Repeating Steps 1-4, we see that there is also a group of vertices $\hat{\omega}_1,\cdots,\hat{\omega}_{\hat{k}}$ such that $\mathcal{T}^{[j]}$ is uncontrollable with $\hat{\omega}_1,\cdots,\hat{\omega}_{\hat{k}}$ taking leaders role. Accordingly, the first assertion holds.

Part II: (*Necessity*) Suppose that the system is uncontrollable. The proof in Step 1 of the Part I shows that the Lalacian has an eigenvector y in the form of (6). Recall that $i\in\{1,\cdots,\delta\}$ is a fixed index. For the simplicity of presentation, we assume $Z_{1,i}$ consists of vertices, say $v_{\tau+1},v_{\tau+2}$, i.e., $Z_{1,i}=\{v_{\tau+1},v_{\tau+2}\}$, where $\tau=N+l-|Z|$. The same reasonings apply to the general situation. Note that the Laplacian is

$$\mathcal{L}=\begin{bmatrix} \mathcal{L}_1 & 0 & \cdots & \cdots & 0 & \mathcal{R}_1 \\ 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \mathcal{L}_i & \ddots & \vdots & \mathcal{R}_i \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & \mathcal{L}_\delta & \mathcal{R}_\delta \\ \mathcal{R}_1^T & \cdots & \mathcal{R}_i^T & \cdots & \mathcal{R}_\delta^T & \mathcal{L}_{\delta+1} \end{bmatrix},$$

and the eigenvector can be conformably partitioned as $y=[\eta_1^T,\cdots,\eta_\delta^T,0^T]^T$, with η_i being nonzero vectors. It follows from the definition of $Z_{1,i}$ that the columns of $\mathcal{R}_i$ are all zero vectors except the first two columns. Since $Z_{1,i}\cap Z_{1,j}=\emptyset$, $i\neq j,\forall j\in\{1,\cdots,\delta\}$, $v_{\tau+1}$ and $v_{\tau+2}$ are not incident vertices of other components $\mathcal{T}_{c_j}$, $j\neq i$, and accordingly the first two columns of $\mathcal{R}_j$ are all zero vectors, $j\neq i, j\in\{1,\cdots,\delta\}$. Recall that $\mathcal{R}_i$ is represented by (8). Then the first two rows of $\begin{bmatrix} \mathcal{R}_1^T & \cdots & \mathcal{R}_i^T & \cdots & \mathcal{R}_\delta^T \end{bmatrix}$ are in the form of

$$\begin{bmatrix} 0 & \cdots & 0 & r_{m_{i-1}+1,\tau+1} & \cdots & r_{m_i,\tau+1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & r_{m_{i-1}+1,\tau+2} & \cdots & r_{m_i,\tau+2} & 0 & \cdots & 0 \end{bmatrix}^T \tag{12}$$

Let

$$\widehat{\mathcal{R}}_i\triangleq\begin{bmatrix} r_{m_{i-1}+1,\tau+1} & r_{m_{i-1}+1,\tau+2} \\ \vdots & \vdots \\ r_{m_i,\tau+1} & r_{m_i,\tau+2} \end{bmatrix},$$

that is, $\widehat{\mathcal{R}}_i$ constitutes of the first two columns of $\mathcal{R}_i$. Consider the following submatrix $\widehat{\mathcal{L}}_i$ of $\mathcal{L}$

$$\widehat{\mathcal{L}}_i\triangleq\begin{bmatrix} \mathcal{L}_i & \widehat{\mathcal{R}}_i \\ \widehat{\mathcal{R}}_i^T & \widehat{\mathcal{L}}_{\delta+1} \end{bmatrix}, \tag{13}$$

where $\widehat{\mathcal{L}}_{\delta+1}$ is the submatrix obtained from $\mathcal{L}_{\delta+1}$ by selecting the first two rows and columns of $\mathcal{L}_{\delta+1}$. It follows from $\mathcal{L}y=\lambda y$ and the specific forms of y and the first two rows of $\begin{bmatrix} \mathcal{R}_1^T & \cdots & \mathcal{R}_i^T & \cdots & \mathcal{R}_\delta^T \end{bmatrix}$ in (12) that

$$\widehat{\mathcal{L}}_i\begin{bmatrix} \eta_i \\ 0 \end{bmatrix}=\lambda\begin{bmatrix} \eta_i \\ 0 \end{bmatrix}, \mathcal{L}_i\eta_i=\lambda\eta_i, \tag{14}$$

that is, $\widehat{\mathcal{L}}_i$ and and $\mathcal{L}_i$ share a common eigenvalue. Since $\widehat{\mathcal{L}}_i$ is the Laplacian matrix corresponding to $\mathcal{T}'_{c_i}$, by Lemma 1, $\mathcal{T}'_{c_i}$ is uncontrollable.

(*Sufficiency*) Note that $\widehat{\mathcal{L}}_i$ expressed by (13) is the Laplacian matrix of subtree $\mathcal{T}'_{c_i}$. Suppose $\mathcal{T}'_{c_i}$ is uncontrollable with leaders selected from $Z_{1,i}$. By Lemma 2, there exists an eigenvector in the form of $[\eta_i,0]^T$ such that the first equation of (14) holds.

Since $Z_{1,i}\cap Z_{1,j}=\emptyset$ for $i\neq j$, $\forall j\in\{1,\cdots,\delta\}$, the other leader vertices in Z are not incident to $\mathcal{T}_{c_i}$. Consequently, one can always find matrices $\widetilde{\mathcal{R}}_i$ and $\overline{\mathcal{L}}_i$ so that

$$\widetilde{\mathcal{L}}_i\triangleq\begin{bmatrix} \mathcal{L}_i & \widehat{\mathcal{R}}_i & 0 \\ \widehat{\mathcal{R}}_i^T & \widehat{\mathcal{L}}_{\delta+1} & \widetilde{\mathcal{R}}_i \\ 0 & \widetilde{\mathcal{R}}_i^T & \overline{\mathcal{L}}_i \end{bmatrix}$$

constitutes the Laplacian matrix of $\mathcal{T}(i)$. Let $\widetilde{\eta}\triangleq\begin{bmatrix} \eta_i^T, & 0, & 0 \end{bmatrix}^T$. It follows from (14) that

$$\widetilde{\mathcal{L}}_i\widetilde{\eta}=\lambda\widetilde{\eta}.$$

Since $\mathcal{L}_i\eta_i=\lambda\eta_i$, $\widetilde{\mathcal{L}}_i$ and $\mathcal{L}_i$ share the common eigenvalue λ. Then, by Lemma 1, $\mathcal{T}(i)$ is uncontrollable. Moreover, by Lemma 4, the system is uncontrollable. ■

In case the leader is single, we assume that the eigenvalues of $\mathcal{F}$ are $\mu_1\leq\cdots\leq\mu_N$, and the eigenvalues of $\mathcal{L}$ are $\lambda_1\leq\cdots\leq\lambda_{N+1}$.

Proposition 2: The multi-agent system is controllable under a single leader v if and only if the eigenvalues of $\mathcal{F}$ strictly interlace those of $\mathcal{L}$, i.e., $\lambda_1<\mu_1<\lambda_2<\cdots<\lambda_N<\mu_N<\lambda_{N+1}$.

Proof: It follows from Cauchy's interlace theorem that

$$\lambda_1\leq\mu_1\leq\lambda_2\leq\cdots\leq\lambda_N\leq\mu_N\leq\lambda_{N+1}. \tag{15}$$

By Lemma 1, the system is controllable if and only if $\mathcal{L}$ and $\mathcal{F}$ do not share any common eigenvalues. This, together with (15), gives the result. ■

Proposition 3: For a multi-agent system with tree interconnection graph, if the system is uncontrollable under a single leader, then

- there is a principle submatrix $\mathcal{F}_i$ in (5), which shares a common eigenvalue with $\mathcal{L}$, and its determinant value is one, i.e., $|\mathcal{F}_i|=1$, $i\in\{1,\cdots,\gamma\}$.
- for principle submatrices $\mathcal{L}_i$ in (7), each $\mathcal{L}_i$ shares a common eigenvalue with $\mathcal{L}$, and for each $\mathcal{L}_i$, $|\mathcal{L}_i|=1$, $i=1,\cdots,\delta$.

Proof: In case of the single leader, Laplacian (5) takes the form

$$\mathcal{L}=\begin{bmatrix} \mathcal{F}_1 & 0 & \cdots & 0 & r_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \mathcal{F}_\gamma & r_\gamma \\ r_1^T & \cdots & \cdots & r_\gamma^T & l_{1,1} \end{bmatrix},$$

where r_i are vectors and $l_{1,1}$ is a scalar. By Lemma 2, the uncontrollability of the system implies that there is an eigenvector y associated with an eigenvalue λ of $\mathcal{L}$, which can be partitioned conformably with $\mathcal{L}$ in the form of $y=[y_1^T,\cdots,y_\gamma^T,0]^T$. Since y is nonzero, it can be assumed that at least one y_i, say y_1, is nonzero. Then, $\mathcal{F}_1y_1=\lambda y_1$

follows from $\mathcal{L}y = \lambda y$, that is, $\mathcal{L}$ and $\mathcal{F}_1$ share the common eigenvalue λ.

On the other hand, since the interconnection graph is a tree, each $\mathcal{G}_{c_i}$ corresponding to $\mathcal{F}_i$ is a subtree and in each $\mathcal{G}_{c_i}$ there is only one vertex incident to the leader vertex v_{N+1}. Accordingly, each r_i is an identity vector, $i = 1, \cdots, \gamma$. We assume without loss of generality that the first element of each r_i is not zero, i.e., 1. It follows that $\mathcal{F}_i = \mathcal{L}(\mathcal{G}_{c_i}) + E_{1,1}$, where $\mathcal{L}(\mathcal{G}_{c_i})$ is the Laplacian matrix of $\mathcal{G}_{c_i}$, and $E_{1,1}$ is the matrix whose only nonzero entry is a one in position $(1, 1)$. Then, $|\mathcal{F}_i| = |\mathcal{L}(\mathcal{G}_{c_i})| + |\mathcal{L}(\mathcal{G}_{c_i})_{11}|$, where $\mathcal{L}(\mathcal{G}_{c_i})_{11}$ is the submatrix of $\mathcal{L}(\mathcal{G}_{c_i})$ obtained by eliminating its first row and column. Since $|\mathcal{L}(\mathcal{G}_{c_i})| = 0$, and by the Matrix-Tree Theorem, $|\mathcal{L}(\mathcal{G}_{c_i})_{11}| = 1$, the result follows.

With respect to $\mathcal{L}_i$ in (7), the proofs of Theorem 1 show that the eigenvector y can be partitioned as $y = [\hat{y}_1^T, \cdots, \hat{y}_\delta^T, 0_{\delta+1}^T]^T$ conformably with $\mathcal{L}$ in (7), where each $\hat{y}_i$ is a nonzero vector, $i = 1, \cdots, \delta$. Then $\mathcal{L}y = \lambda y$ leads to $\mathcal{L}_i \hat{y}_i = \lambda \hat{y}_i$ for each $i = 1, \cdots, \delta$. So each $\mathcal{L}_i$ shares a common eigenvalue with $\mathcal{L}$. Following the same lines of arguments as the first assertion, we see that $|\mathcal{L}_i| = 1$, for each $i = 1, \cdots, \delta$. ■

IV. CONCLUSIONS

Controllability of multi-agent systems has been given special attention lately. The multi-agent controllability concept contributes to understanding the mechanisms of effective leadership for leader agents. Also, it is a significant approach for formation control. How the controllability/uncontrollability is affected by the interconnection graph topologies is a central problem to answer. In this paper, we cope with this problem with respect to tree interconnection graph. The uncontrollability of the tree graph is studied via a decomposition of the whole graph. It is shown that the uncontrollability of the whole graph leads to various uncontrollable subgraphs. The result is proved via a constructive approach. In addition, the uncontrollability under single and multiple leaders, as well as the relationship between controllability and the common eigenvalues of components are discussed. Also, a necessary and sufficient condition on controllability is obtained for tree topology. The results add new understandings to the controllability/uncontrollability of multi-agent systems from the graphic point of view.

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