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A Perspective on Reachability and Controllability of Controlled Switched Linear Systems*

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Abstract: This paper presents a unified perspective for geometric and algebraic criteria for the reachability and controllability of switched linear systems. The direct connection between the geometric and algebraic criterion is established as well as that between the subspace algorithm for controllability/reachability and other algebraic criterion. In particular, we investigate the reachability realization problem for a class of controlled switched linear discrete-time systems. Both aperiodic and periodic switching sequences are designed to solve the problem.

Key Words: Switched systems, Controllability, Reachability, Switching sequence

1 INTRODUCTION

Switched systems are referred to as control systems that consist of a finite number of subsystems and a logical rule that orchestrates switchings among them. The study of such systems is significant from both practical and theoretical point of view. A challenging topic in switched systems is to evaluate the effect of switched control on the system operation. It is usually formulated as the controllability problem of switched systems [2, 14, 15]. For switched systems the study of controllability and reachability is difficult because of the involvement of both continuous and discrete dynamics as well as continuous and discrete controls. Since both the control input and the switching sequence are design variables, the reachability and controllability cannot be investigated thoroughly without considering the role of switching sequences. This motivates the study of switching mechanism in the controllability and reachability analysis [3-7]. Most results along this line consist in how to find a switching sequence in a constructive way with its reachable/controllable state set equaling the overall reachable/controllable set of the system. Here we call it the *reachability/controllability realization* problem. It should be noted that a switched linear system is essentially a nonlinear one with switching functions acting as controls. The controllability of switching systems deserve further study from the general nonlinear control point of view [8]. Most results on controllability and reachability of switched linear systems were expressed in terms of geometric symbols (e.g. [1, 3, 7, 9-11]), while a few others algebraic (e.g. [2, 12]). The geometric criteria have the advantage of a straightforward characterization of the reachable/controllable subspace, while the algebraic criteria can be checked and manipulated more conveniently. It is worth noting that there is a lack of systematic perspective on the connections between these two kinds of results as well as the related subspace-based algorithms. In this paper, we present not only these connections but also some improved geometric and algebraic criteria. Also the relationship between the existing subspace-based algorithms is revealed.

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2 DEFINITIONS AND SUPPORTING LEMMAS

A switched linear discrete-time system is described by

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^p$ the input, $\sigma(k) : \{0, 1, \cdots\} \to \Lambda := \{1, \cdots, m\}$ is the switching path to be designed.Throughout the paper, we assume that the discrete-time switched system (1) is *reversible* [10]. A natural number k is said to be a *switching instant* of the switching sequence if $\sigma(k) \neq \sigma(k-1)$. Let $t_1, t_2, \cdots, t_s$ be the ordered switching instants with $0 < t_1 < t_2 < \cdots < t_s$. The sequence $0, t_1, \cdots, t_s$ is said to be the *switching instant sequence* of σ. A switching sequence π with length $L(\pi) = s$ is denoted by $\pi = \{(i_0, h_0) \cdots (i_{s-1}h_{s-1})\}$, where $0, h_0, h_0 + h_1, \cdots, \sum_{j=0}^{s-1} h_j$ is the switching instant sequence, $i_0 = \sigma(0), i_1 = \sigma(h_0)$, $i_{s-1} = \sigma(\sum_{j=0}^{s-2} h_j)$ is the *switching index sequence*. Set $\underline{k} = \{0, \cdots, k-1\}$. We recall some definitions [10, 11].

Definition 1 State x is reachable, if there exist a time instant $k > 0$, a switching path $\sigma : \underline{k} \to \Lambda$, and inputs $u : \underline{k} \to \mathbb{R}^n$, such that $x(0) = 0$, and $x(k) = x$. The reachable set of system (1) is the set of states which are reachable. System (1) is said to be (completely) reachable, if its reachable set is $\mathbb{R}^n$. The controllability counterpart can be given by replacing '$x(0) = 0$, and $x(k) = x$' with '$x(0) = x$ and $x(k) = 0$'.

Definition 2 Given a switching sequence $\pi = \{(i_0, h_0) \cdots (i_{s-1}, h_{s-1})\}$, the reachable state set of π is defined by $\mathscr{T}(\pi) = \{x | \exists \text{inputs } u(i), i \in \underline{s}, \text{ such that } x(0) = 0 \text{ and } x(\sum_{j=0}^{s-1} h_j) = x\}$.

Definition 3 Given a switching sequence $\pi = \{(i_0, h_0) \cdots (i_{s-1}, h_{s-1})\}$, the controllable state set of π is defined by $\mathcal{C}(\pi) = \{x | \exists \text{ inputs } u(i), i \in \underline{s}, \text{ such that } x(0) = x \text{ and } x(\sum_{j=0}^{s-1} h_j) = 0\}$.

Lemma 1 [4] Given a switching sequence $\pi = \{(i_0, h_0) \cdots (i_{s-1}, h_{s-1})\}$, the reachable state set of π is a linear space described by

$$\begin{aligned}\mathscr{T}(\pi) = & A_{i_{s-1}}^{h_{s-1}} \cdots A_{i_1}^{h_1} \operatorname{Im}\left[B_{i_0}, A_{i_0}B_{i_0}, \cdots, A_{i_0}^{h_0-1}B_{i_0}\right] + \cdots \\ & + A_{i_{s-1}}^{h_{s-1}} \operatorname{Im}\left[B_{i_{s-2}}, A_{i_{s-2}}B_{i_{s-2}}, \cdots, A_{i_{s-2}}^{h_{s-2}-1}B_{i_{s-2}}\right] \\ & + \operatorname{Im}\left[B_{i_{s-1}}, A_{i_{s-1}}B_{i_{s-1}}, \cdots, A_{i_{s-1}}^{h_{s-1}-1}B_{i_{s-1}}\right]\end{aligned}$$

where $\operatorname{Im}(\cdot)$ denotes the image space of a matrix.

We denote by $\mathscr{T}$ and $\mathscr{C}$ the set of all reachable and controllable states of systems (1), respectively. Clearly, $\mathscr{T} = \bigcup_{\forall\pi} \mathscr{T}(\pi)$. Given a switching sequence $\pi = \{(i_0, h_0) \cdots (i_{s-1}, h_{s-1})\}$, denote $A_\pi = \prod_{j=s-1}^{0} A_{i_j}^{h_j}$, where the product notation is to be read left-to-right, i.e., $\prod_{j=1}^{s} X_j = X_1 X_2 \cdots X_s$. The following result is straightforward from Lemma 1.

Lemma 2 [11] Given switching sequences π_1 and π_2, $\mathscr{T}(\pi_1 \wedge \pi_2) = A_{\pi_2}\mathscr{T}(\pi_1) + \mathscr{T}(\pi_2)$.

Given $A \in \mathbb{R}^{n\times n}$, and a linear subspace $\mathscr{W} \subseteq \mathbb{R}^n$, we denote $\langle A|\mathscr{W}\rangle = \sum_{i=1}^{n} A^{i-1}\mathscr{W}$. $\langle A|\mathscr{W}\rangle$ is a minimum A-invariant subspace that contains $\mathscr{W}$. Define the following subspace sequence $\mathscr{P}_j = \sum_{i=1}^{j} A^{i-1}\mathscr{W}, j = 1, 2, \cdots$. Clearly $\langle A|\mathscr{W}\rangle = \mathscr{P}_n$. Let ϑ be the integer such that $\vartheta = \min\{j \mid \mathscr{P}_j = \mathscr{P}_{j+1},\ j = 1, 2, \cdots\}$. In association with A, we denote by ρ the degree of its minimal polynomial.

Lemma 3 [13] Given a matrix $A \in \mathbb{R}^{n\times n}$, and a linear subspace $\mathscr{W} \subseteq \mathbb{R}^n$,

$$\mathscr{P}_j = \mathscr{P}_\vartheta, \quad \text{for all } j \geqslant \vartheta$$

with ϑ satisfying $\vartheta \leqslant \min\{n - \dim\mathscr{W} + 1, \rho\}$.

Remark 1 An immediate consequence of this lemma is $\langle A|\mathscr{W}\rangle = \sum_{i=1}^{\vartheta} A^{i-1}\mathscr{W}$. For the convenience of statement, we hereafter call ϑ the **$(A, \mathscr{W})$-invariant subspace index**.

Lemma 4 [11] Given a switching sequence π, $\mathscr{T}(\pi^{\wedge n}) = \langle A_\pi|\mathscr{T}(\pi)\rangle$.

Combining this lemma with the property of invariant subspace and the non-singularity of A_i, $i \in \Lambda$, yields

$$A_{\pi^{\wedge s}}\mathscr{T}(\pi^{\wedge n}) = \mathscr{T}(\pi^{\wedge n}), \quad \forall s = 1, 2, \cdots \tag{2}$$

For any matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$, set $\mathcal{B} := \operatorname{Im} B$. We need the following subspace sequence [3, 10].

$$\mathscr{V}_1 = \sum_{s=1}^{m} \mathcal{B}_s, \mathscr{V}_i = \sum_{s=1}^{m} \langle A_s|\mathscr{V}_{i-1}\rangle, i = 2, 3, \cdots \tag{3}$$

Define $\mathscr{V} = \sum_{i=1}^{\infty} \mathscr{V}_i$. It follows that $\mathscr{T}(\pi) \subseteq \mathscr{V}$. This yields $\mathscr{T} \subseteq \mathscr{V}$. Furthermore, the following result holds.

Lemma 5 [10, 11] For discrete-time switched linear systems (1), $\mathscr{T} = \mathscr{V} = \mathcal{C}$.

Let $\mu = \min\{i |\ \mathscr{V}_i = \mathscr{V}_{i+1},\ i = 1, 2, \cdots\}$. It can be readily seen that $\mathscr{V}_1 \subset \mathscr{V}_2 \subset \cdots \subset \mathscr{V}_\mu$, and $\mathscr{V} = \mathscr{V}_\mu$. Obviously, μ is fixed once the switched system (1) is given. Furthermore $\mu \leqslant n - d_1 + 1$. Hereafter, we call μ the **joint invariant subspace index of** $(A_1, \cdots, A_m;\ B_1, \cdots, B_m)$, and denote by d_i the dimension of $\mathscr{V}_i$, i.e., $d_i = \dim\mathscr{V}_i,\ i = 1, 2, \cdots, \mu$.

3 A UNIFIED PERSPECTIVE ON REACHABILITY AND CONTROLLABILITY CRITERIA

3.1 Geometric and Algebraic Criteria

Let $\omega_{i,j},\ i, j = 1, \cdots, m$, be the $(A_i, \mathcal{B}_j)$-invariant subspace index, and $\vartheta_{i,j},\ i = 1, \cdots, m; j = 1, \cdots, \mu - 1$, the $(A_i, \mathscr{V}_j)$-invariant subspace index. Denote by ρ_i the degree of the minimal polynomial of A_i. It follows from Lemma 3 that $\mathscr{V}_i$ can be written in the form

$$\mathscr{V}_i = \sum_{s=1}^{m} \sum_{j=1}^{\vartheta_{s,i-1}} A_s^{j-1}\mathscr{V}_{i-1},\ i = 2, 3, \cdots, \mu \tag{4}$$

with $\vartheta_{s,i-1}$, the $(A_s, \mathscr{V}_{i-1})$-invariant subspace index,

$$\vartheta_{s,i-1} \leqslant \min\{n - d_{i-1} + 1, \rho_s\}, s = 1, \cdots, m \tag{5}$$

$i = 2, \cdots, \mu$, and $\omega_{i,j}$ satisfying

$$\omega_{i,j} \leqslant \min\{n - \dim\mathcal{B}_j + 1, \rho_i\} \tag{6}$$

Set $\vartheta_i = \max\{\omega_{i,s}, \vartheta_{i,j}; s = 1, \cdots, m; j = 1, \cdots, \mu - 1\}, i = 1, \cdots, m$, and denote $\beta \stackrel{\Delta}{=} \min\{\dim\mathcal{B}_j, j = 1, \cdots, m\}$, $\rho \stackrel{\Delta}{=} \max\{\rho_s,\ s = 1, \cdots, m\}$. By (5), (6), and the inequality $1 \leqslant \beta \leqslant d_1 < d_2 < \cdots < d_{\mu-1} < d_\mu = \dim\mathscr{V}$, one has $\vartheta_i \leqslant \min\{n - \beta + 1, \rho\}, i = 1, \cdots, m$. Accordingly, $n - \vartheta_i \geqslant 0$ and one can introduce the following definition $\Delta(\vartheta_i + j) \stackrel{\Delta}{=} \{0, 1, \cdots, \vartheta_i - 1 + j \mid j = 0, 1, \cdots, n - \vartheta_i\}, i = 1, \cdots, m$. Note that $\Delta(\vartheta_i) \stackrel{\Delta}{=} \{0, 1, \cdots, \vartheta_i - 1\}, i = 1, \cdots, m$. Thus we associate with each subsystem matrix A_i a nonnegative integer set $\Delta(\vartheta_i)$, which is the smallest one in the sense that $\Delta(\vartheta_i) \subset \Delta(\vartheta_i + j),\ j = 1, 2, \cdots, n - \vartheta_i$. To proceed, we need to define the following subspaces

$$\mathfrak{M}\left(l_{i_1}, \cdots, l_{i_{\mu-1}}\right) \stackrel{\Delta}{=} \sum_{i_0, \cdots, i_{\mu-1} \in \Lambda}^{j_k \in \Delta(\vartheta_{i_k} + l_{i_k}), k=1, \cdots, \mu-1} A_{i_{\mu-1}}^{j_{\mu-1}} \cdots A_{i_1}^{j_1}\mathcal{B}_{i_0} \tag{7}$$

where $l_{i_s} \in \{1, 2, \cdots, n - \vartheta_{i_s}\}$; $s = 1, \cdots, \mu - 1$. In particular, when $l_{i_1} = \cdots = l_{i_{\mu-1}} = 0$, expression (7) becomes to

$$\mathfrak{M} \stackrel{\Delta}{=} \sum_{i_0, \cdots, i_{\mu-1} \in \Lambda}^{j_1 \in \Delta(\vartheta_{i_1}), \cdots, j_{\mu-1} \in \Delta(\vartheta_{i_{\mu-1}})} A_{i_{\mu-1}}^{j_{\mu-1}} \cdots A_{i_1}^{j_1}\mathcal{B}_{i_0} \tag{8}$$

Since $\Delta(\vartheta_{i_s}) \subset \Delta(\vartheta_{i_s} + l_{i_s})$, it follows from (7) and (8) that for an arbitrary $l_{i_s} \in \{1, 2, \cdots, n - \vartheta_{i_s}\}$; $\mathfrak{M} \subseteq \mathfrak{M}\left(l_{i_1}, \cdots, l_{i_{\mu-1}}\right)$, which means the inclusion is true for an arbitrary group of integers $\{l_{i_1}, \cdots, l_{i_{\mu-1}}\}$ with at least one nonzero element.Finally, combining (3), (4) with (8) yields

$$\mathscr{V}_\mu = \sum_{i_0, \cdots, i_{\mu-1} \in \Lambda}^{j_k \in \Delta(\vartheta_{i_k}), k=1, \cdots, \mu-1} A_{i_{\mu-1}}^{j_{\mu-1}} \cdots A_{i_1}^{j_1}\mathcal{B}_{i_0} = \mathfrak{M} \tag{9}$$

Since $\mathscr{V} = \mathscr{V}_\mu$, by Lemma 5, we have the following result.

Theorem 1 The switched linear discrete-time system (1) is reachable if and only if

$$\mathfrak{M} = \mathbb{R}^n$$

Remark 2 The contribution of Theorem 1 consists in providing a simplified geometric characterization for the reachability subspace $\mathscr{T}$, i.e. $\mathscr{T} = \mathfrak{M}$. More specifically, $\mathscr{T}$ was written in [1] in the form

$$\mathscr{T} = \sum_{i_0,\cdots,i_{n-1}\in\Lambda}^{j_1,\cdots,j_{n-1}\in\{0,\cdots,n-1\}} A_{i_{n-1}}^{j_{n-1}} \cdots A_{i_1}^{j_1} \mathcal{B}_{i_0} \tag{10}$$

which clearly corresponds in (7) to the case $\mu = n$ with $l_{i_s} = n - \vartheta_{i_s}, s = 1, \cdots, n$. The difference between (8) and (10) lie in (a): The number of multiplying matrices involved in each adding term in (8) is $\mu (\leqslant n - d_1 + 1)$ which is not greater than n, the same kind of number in (10) as μ in (8). Consequently, the number of adding terms in (10) is greatly reduced in (8), especially when μ is much less than n. (b): The maximum number of power in association with each multiplying system matrix A_{i_s} in (10) is $n-1$, which is reduced to $\vartheta_{i_s} - 1$ in (8), $i_s \in \Lambda$. Note that by (5) and (6) $\vartheta_{i_s} \leqslant \min\{n - \dim \mathcal{B}_{i_s} + 1, \rho_{i_s}\}$.

Next we demonstrate the corresponding algebraic criteria for Theorem 1. Let $i_1, \cdots, i_{\mu-1} \in \Lambda$ be given. Define at first the following m matrices for $s = 1, \cdots, m$,

$$\begin{aligned} &\text{Œ}^{\mu-1}(s, i_1, \cdots, i_{\mu-1}) \\ &\triangleq \left[A_{i_{\mu-1}}^{j_{\mu-1}} \cdots A_{i_1}^{j_1} B_s\right]_{j_1\in\Delta(\vartheta_{i_1}),\cdots,j_{\mu-1}\in\Delta(\vartheta_{i_{\mu-1}})} \end{aligned} \tag{11}$$

Then define

$$\mathfrak{A}^{\mu-1}(s) \triangleq \left[\text{Œ}^{\mu-1}(s, i_1, \cdots, i_{\mu-1})\right]_{i_1,\cdots,i_{\mu-1}\in\Lambda} \tag{12}$$

Let

$$\mathbb{M} = \left[\mathfrak{A}^{\mu-1}(1)\, \mathfrak{A}^{\mu-1}(2) \cdots \mathfrak{A}^{\mu-1}(m)\right] \tag{13}$$

We have the following Kalman-type rank criterion for reachability.

Theorem 2 The switched linear discrete-time system (1) is reachable if and only if the controllable matrix $\mathbb{M}$ is of full row rank, i.e.

$$rank\, \mathbb{M} = n$$

Proof From (9), $\mathscr{V}_\mu$ can be written in the form

$$\begin{aligned} \mathscr{V}_\mu = &\sum_{i_1,\cdots,i_{\mu-1}\in\Lambda} \sum_{j_k\in\Delta(\vartheta_{i_k}),k=1,\cdots,\mu-1} A_{i_{\mu-1}}^{j_{\mu-1}} \cdots A_{i_1}^{j_1} \mathcal{B}_1 + \cdots \\ + &\sum_{i_1,\cdots,i_{\mu-1}\in\Lambda} \sum_{j_k\in\Delta(\vartheta_{i_k}),k=1,\cdots,\mu-1} A_{i_{\mu-1}}^{j_{\mu-1}} \cdots A_{i_1}^{j_1} \mathcal{B}_m \end{aligned} \tag{14}$$

By (11), for a group of given $i_1, \cdots, i_{\mu-1}$,

$$\begin{aligned} &\text{Im}\, \text{Œ}^{\mu-1}(s, i_1, \cdots, i_{\mu-1}) \\ &= \sum_{j_k\in\Delta(\vartheta_{i_k}),k=1,\cdots,\mu-1} A_{i_{\mu-1}}^{j_{\mu-1}} \cdots A_{i_1}^{j_1} \mathcal{B}_s, s = 1, \cdots, m \end{aligned}$$

Furthermore, it follows from (12) that for $s = 1, \cdots, m$

$$\begin{aligned} &\text{Im}\, \mathfrak{A}^{\mu-1}(s) \\ &= \sum_{i_1,\cdots,i_{\mu-1}\in\Lambda} \sum_{j_k\in\Delta(\vartheta_{i_k}),k=1,\cdots,\mu-1} A_{i_{\mu-1}}^{j_{\mu-1}} \cdots A_{i_1}^{j_1} \mathcal{B}_s \end{aligned}$$

Combining this with (13) and (14) yields $\text{Im}\, \mathbb{M} = \mathscr{V}_\mu$. The result then follows from Lemma 5.

The algebraic conditions on controllability were recently studied in [2, 12] by employing a concept of *joint controllability matrices*. We revisit this concept. Define

$$\text{æ}^k(i_1, \cdots, i_k) \triangleq \left[A_{i_k}^{j_k} \cdots A_{i_2}^{j_2} A_{i_1}^{j_1} B_{i_1}\right]_{j_1,\cdots,j_k\in\{0,1,\cdots,n-1\}}$$

Then let $\text{Æ}^0(i) = \text{æ}^1(i), \cdots, \text{Æ}^k(i) = [\text{æ}^{k+1}(i, i_1, \cdots, i_k)]_{i_1,\cdots,i_k\in\Lambda}$. The joint controllability matrices can be iteratively defined as $W^0 = [\text{Æ}^0(1)\text{Æ}^0(2) \cdots \text{Æ}^0(m)], \cdots, W^k = [\text{Æ}^k(1)\text{Æ}^k(2) \cdots \text{Æ}^k(m)]$. There exists a *joint controllability coefficient* k_r of the system defined by $k_r = \arg\min_l \{\text{rank}\, W^l = \text{rank}\, W^{l+1}\}$ [2]. Yang proved that a necessary condition for the controllability is $rank\, W^{k_r} = n$. Then Stikkel, Bokor and Szabó showed that this condition is also necessary *provided that the switching signal is exciting.* So the algebraic criterion on controllability has not been solved completely. In particular, there are two questions related to it, which are: (a) Few properties are known on k_r, especially the exact value. So we want to know whether there are any other characterizations for k_r. (b) Whether can the exciting assumption on switching signals be removed? There is a positive answer for the second question in Theorem 2. To analyze the first one, we present a modified version of joint controllability matrices. Let $\text{Œ}^0(i) \triangleq B_i$, $i = 1, \cdots, m$. $\text{Œ}^k(i, i_1, \cdots, i_k) \triangleq \left[A_{i_k}^{j_k} \cdots A_{i_1}^{j_1} B_i\right]_{j_1\in\Delta(\vartheta_{i_1}),\cdots,j_k\in\Delta(\vartheta_{i_k})}$. Define $\mathfrak{A}^0(i) = \text{Œ}^0(i)$ $, \cdots,$ $\mathfrak{A}^k(i) = \left[\text{Œ}^k(i, i_1, \cdots, i_k)\right]_{i_1,\cdots,i_k\in\Lambda}$, and $W^0 = [\mathfrak{A}^0(1)\, \mathfrak{A}^0(2) \cdots \mathfrak{A}^0(m)], \cdots, W^k = \left[\mathfrak{A}^k(1)\, \mathfrak{A}^k(2) \cdots \mathfrak{A}^k(m)\right]$. It can be seen that the joint controllability matrices defined in this way has the property: $\text{Im} W^k = \mathscr{V}_{k+1}$, $k = 0, 1, 2, \ldots$. An immediate consequence of this observation is the following result.

Theorem 3 The relationship between the joint controllability coefficient k_r and the joint invariant subspace index of system (1) is

$$k_r = \mu - 1$$

Remark 3 Theorems 1 and 2 exhibit a direct connection and correspondence between the geometric and algebraic criteria. Theorems 1-3 not only present simplified geometric and algebraic criteria for controllability and reachability of switched linear systems, but also demonstrate these two criteria in a systematic and unified way for the first time. At the same time the relationship between the joint controllability coefficient and the joint invariant subspace index is revealed.

3.2 Computational Issues and Other Algebraic Rank Condition

$$\mathscr{W}_0 = \sum_{j=1}^m \mathcal{B}_j, \qquad \mathscr{W}_{k+1} = \mathscr{W}_0 + \sum_{j=1}^m A_j \mathscr{W}_k \tag{15}$$

Let $\mathscr{W}^* = \lim_{k\to\infty} \mathscr{W}_k$, it is proved in [12] that $\text{Im} W^{k_r} = \mathscr{W}^*$. With respect to the subspace sequence (15), we have

Proposition 1 The subspace sequence (15) can be equivalently written as

$$\mathscr{W}_0 = \sum_{j=1}^{m} \mathcal{B}_j, \qquad \mathscr{W}_{k+1} = \mathscr{W}_k + \sum_{j=1}^{m} A_j \mathscr{W}_k \tag{16}$$

As a consequence, if a nonnegative number γ is defined by $\gamma = \min\{k \mid \mathscr{W}_k = \mathscr{W}_{k+1}, k = 0, 1, \cdots\}$, then

$$\mathscr{W}_0 \subset \mathscr{W}_1 \subset \cdots \subset \mathscr{W}_\gamma = \mathscr{W}_{\gamma+1} = \cdots = \mathscr{W}^* = \mathscr{T} \tag{17}$$

Proof We show it by induction. Clearly $\mathscr{W}_1 = \mathscr{W}_0 + \sum_{j=1}^{m} A_j \mathscr{W}_0$. Suppose $\mathscr{W}_{k+1} = \mathscr{W}_k + \sum_{j=1}^{m} A_j \mathscr{W}_k$. Then

$$\begin{aligned}
\mathscr{W}_{k+2} &= \mathscr{W}_0 + \sum_{j=1}^{m} A_j \mathscr{W}_{k+1} = \mathscr{W}_0 + \sum_{j=1}^{m} A_j \left(\mathscr{W}_k + \sum_{l=1}^{m} A_l \mathscr{W}_k \right) \\
&= \mathscr{W}_0 + \sum_{j=1}^{m} \left(A_j \mathscr{W}_k + A_j \sum_{l=1}^{m} A_l \mathscr{W}_k \right) \\
&= \left(\mathscr{W}_0 + \sum_{j=1}^{m} A_j \mathscr{W}_k \right) + \sum_{j=1}^{m} A_j \left(\mathscr{W}_k + \sum_{l=1}^{m} A_l \mathscr{W}_k \right) \\
&= \mathscr{W}_{k+1} + \sum_{j=1}^{m} A_j \mathscr{W}_{k+1}
\end{aligned}$$

That is (16) holds. By (16) and the definition of γ, one has $\mathscr{W}_0 \subset \mathscr{W}_1 \subset \cdots \subset \mathscr{W}_\gamma = \mathscr{W}_{\gamma+1} = \cdots = \mathscr{W}^*$. The equality $\mathscr{W}^* = \mathscr{T}$ follows by combing (10), Lemma 5 and the proof of Proposition 1 in [12].

Remark 4 The subspace sequence (16) is exactly the one used by Sun et al. in [3]. Proposition 1 tells us that the subspace sequences (15) and (16) are actually equivalent from each other. The advantage of (15) lies in its simple form while the subspace (16) possesses good property (17). By Proposition 1, these two advantages can be combined together when one want to calculate the reachable/controllable subspace for switched linear systems. In other words, one can start computing $\mathscr{W}_k, k = 0, 1, 2 \cdots$, according to (15) which is simpler than (16), and stop the algorithm at most within $n - \dim \mathscr{W}_0$ steps because according to (17), $\gamma \leqslant n - \dim \mathscr{W}_0$.

Next we state another algebraic criterion. Let

$$\begin{aligned}
\Gamma \triangleq [&B_1, \cdots, B_m, A_1 B_1, \cdots, A_1 B_m, \cdots, A_m B_1, \cdots, A_m B_m, \\
&\cdots, A_1^\gamma B_1, \cdots, A_1^\gamma B_m, A_1^{\gamma-1} A_2 B_1, \cdots, A_1^{\gamma-1} A_2 B_m, \cdots, \\
&A_m^\gamma B_1, \cdots, A_m^\gamma B_m]
\end{aligned}$$

Theorem 4 The switched linear discrete-time system (1) is reachable if and only if the matrix Γ is of full row rank, i.e. $\operatorname{rank} \Gamma = n$.

The proof is omitted due to the space limitation.

4 REACHABILITY REALIZATION FOR A CLASS OF CONTROLLED SWITCHED LINEAR DISCRETE-TIME SYSTEMS

Lemma 6 Given matrices B_1, B_2; if $A_1, \cdots, A_t$ are sufficiently close to the identity matrix I_n, then for an arbitrary group of $i_1, \cdots, i_s \in \{1, \cdots, t\}$, it satisfies

$$\operatorname{rank}[B_1, A_1 \cdots A_t B_2] \geqslant \operatorname{rank}[B_1, A_{i_1} A_{i_2} \cdots A_{i_s} B_2]. \tag{18}$$

Denote $\beta := \max_{i \in \Lambda}\{\dim \mathcal{B}_i\}$. The following lemma is required for the derivation of main result in this section.

Lemma 7 [5] With respect to the system (1), $\sum_{i=1}^{\mu} \eta_i$ switching sequences $\pi_{1,1}, \cdots, \pi_{1,\eta_1}$, $\pi_{2,1}, \cdots, \pi_{2,\eta_2}$, $\cdots$, $\pi_{\mu,1}, \cdots, \pi_{\mu,\eta_\mu}$ can be designed so that $\mathscr{T} = \mathscr{V}_\mu = \sum_{i=1}^{\mu} \sum_{j=1}^{\eta_i} \mathscr{T}(\pi_{i,j})$ with $\eta_1, \cdots, \eta_\mu$ satisfying $\eta_1 \leqslant \min\{m, d_1 - \beta + 1\}$; $1 \leqslant \eta_i \leqslant d_i - d_{i-1}, i = 2, \cdots, \mu$; and the length $L(\pi_{i,j})$ of switching sequence $\pi_{i,j}$ satisfying $L(\pi_{i,1}) = \cdots = L(\pi_{i,\eta_i}) = i, i = 1, \cdots, \mu$.
Denote $\gamma_1 = d_1,\ \gamma_k = d_k - d_{k-1},\ k = 2, 3, \cdots, \mu$.

Theorem 5 For switched linear discrete-time system (1) with A_i sufficiently close to the identity matrix, a single switching sequence $\tilde{\pi}$ can be designed such that

$$\mathscr{T}(\tilde{\pi}) = \mathscr{V}_\mu = \mathscr{T} \tag{19}$$

where the length of $\tilde{\pi}$ satisfies $\frac{\mu(\mu+1)}{2} \leqslant L(\tilde{\pi}) \leqslant \sum_{k=1}^{\mu} k\gamma_k - \beta + 1$.

Denote $\pi_m = \{(i_0, h_0) \cdots (i_{m-1}, h_{m-1})\}$ with $i_0 = 1, \cdots, i_{m-1} = m$. Then $\pi_m^{\wedge k} = \underbrace{\pi_m \wedge \cdots \wedge \pi_m}_{k \text{ times}}$ is a cyclic switching sequence with k switching periods, and the number of switchings involved in it is km.

Lemma 8 For positive integer k, if $k < n$, then $\mathscr{T}(\pi_m^{\wedge k}) \subseteq \mathscr{T}(\pi_m^{\wedge n})$. Otherwise, $\mathscr{T}(\pi_m^{\wedge k}) = \mathscr{T}(\pi_m^{\wedge n}), \quad \forall k \geqslant n$.

Theorem 6 For switched linear discrete-time system (1) with $A_1, \cdots, A_m$ sufficiently close to the identity matrix I_n, it holds that $\mathscr{T}(\pi_m^{\wedge n}) = \mathscr{V}$.

5 CONCLUSION

The work presented in this paper contributes to the field by providing a unified perspective for the controllability and reachability algebraic and geometric criteria as well as the corresponding subspace based algorithms. Some connections between the algebraic and geometric criteria are revealed as well as that between the algorithms and the rank conditions. We also studied the reachability realization problem for a class of switched linear discrete-time systems. Both aperiodic and periodic switching sequences are designed to solve the problem.

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