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Hybrid $\mathcal{H}_\infty$ State Feedback Control for Discrete-Time Switched Linear Systems

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Abstract— **In this paper, the co-design of continuous-variable controllers and discrete-event switching logics, both in state feedback form, for discrete-time switched linear control systems is investigated. Sufficient synthesis conditions for this co-design problem are proposed here in the form of bilinear matrix inequalities, which is based on the argument of multiple Lyapunov functions. The closed-loop switched system forms a special class of piecewise linear hybrid systems, and is shown to be exponentially stable with a finite l_2 induced gain.**

Index Terms— **Switched systems, controller synthesis, l_2 induced gain, Lyapunov methods.**

I. Introduction

A remarkable feature of a switched system is that even when all its subsystems are unstable it may still be possible to stabilize it by properly designed switching laws [12], [6], [15]. The synthesis of stabilizing switching signals for a given collection of dynamical systems, especially linear systems, has attracted a lot of attention recently; see for example the survey papers [12], [16], [6], [13], the recent books [11], [23] and the references cited therein.

Early efforts along this direction were focused on quadratic stabilization for certain classes of systems. For example, a quadratic stabilization switching law between two linear time invariant (LTI) systems was considered in [25], and it was shown that the existence of a stable convex combination of the two subsystem matrices implies the existence of a quadratic Lyapunov function and a state-dependent switching rule that (quadratically) stabilizes the switched system. A generalization to more than two LTI subsystems was suggested in [20] by using a “min-projection strategy”. In [8], it was shown that the stable convex combination condition is also necessary for the quadratic stabilizability of two mode switched LTI system. However, it is only sufficient for switched LTI systems with more than two modes. A necessary and sufficient condition for quadratic stabilizability of switched controller systems was derived in [22]. There are extensions of [25] to the output-dependent switching and discrete-time cases [12], [27]. For robust stabilization, a quadratic stabilizing switching law was designed for polytopic uncertain switched linear systems based on linear matrix inequality (LMI) techniques in [27]. All of these methods guarantee stability by using a common quadratic Lyapunov function, which is conservative in the sense that there are switched systems that can be asymptotically (or exponentially) stabilized without using a common quadratic Lyapunov function.

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There have been some results in the literature that propose constructive synthesis methods to switched systems using multiple Lyapunov functions. For example, in [24], piecewise quadratic Lyapunov functions was employed for two mode switched LTI systems. Exponential stabilization for continuous-time switched LTI systems was considered in [18] also based on piecewise quadratic Lyapunov functions, and the synthesis problem was formulated as a bilinear matrix inequality (BMI) problem. In [10], a probabilistic algorithm was proposed for the synthesis of an asymptotically stabilizing switching law for switched LTI systems along with a piecewise quadratic Lyapunov function. A necessary and sufficient condition for asymptotically stabilizability of continuous-time switched linear systems was proposed in [15]. There are also some interesting work on designing the state-feedback or output feedback gains for each subsystem so as to stabilize the switched system under arbitrary switching [5], [7], under given switching signals (e.g. slow switching [4]), or under autonomous switchings due to the partition of the state space [17], [21]. However, it is rare to design the continuous controllers and switching logics together, which is so-called co-design problem. The main challenge is due to the coupling between the continuous controllers and switching logics.

The co-design problem for a class of continuous-time switched LTI systems was considered in [19], where BMI synthesis condition is developed for exponential stabilization. The first part of this current paper can be seen as an extension of [19] to the discrete-time counterpart. However, the extension is nontrivial since the switching instants for the discrete-time case cannot be simply captured as the time instants when the state trajectories cross the switching surfaces. In the second part of the paper, we studied the switching controller synthesis problem to guarantee that the l_2 induced gain is below certain bound. Most of the existing results on the robust performances of switched systems are primarily on the performance analysis [26], [9] or on the continuous feedback controllers design [17], while switching controller synthesis and co-design are still lacking.

Our focus here is the co-design of switching signals and state feedback gains. Some preliminary results of this paper appeared in [14], where stabilizing switching signals are synthesized. The rest of the paper is organized as follows. After formulating the co-design problem in Section II, Section III characterizes the stabilizing switching signals based on the MLF theorem. The stabilization co-design problem is investigated in Section IV, while the co-design problem to achieve finite l_2 induced gain is studied in Section V, which is based on an extension of the MLF theorem. Sufficient conditions for controller synthesis areproposed in the form of BMIs. Finally, concluding remarks are presented.

Notation: The relation $A > B$ ($A < B$) means that the matrix $A - B$ is positive (negative) definite. The superscript T stands for matrix transposition and the matrix I stands for identity matrix of proper dimension. l_2 is the Lebesgue space consisting of all discrete-time vector-valued function that are square-summable over $\mathbb{Z}^+$. $\|z\|_2$ denotes the l_2 norm of z, which is defined as $\|z\|_2^2 = \sum_0^{+\infty} z^T(t)z(t)$.

II. Problem Formulation

In this paper, we consider a collection of discrete-time linear control systems described by the difference equations

$$\begin{cases} x(t+1) &= A_i x(t) + B_i u(t) + B_i^w w(t) \\ z(t) &= C_i x(t) + D_i u(t) + D_i^w w(t), \end{cases} \tag{1}$$

where $t \in \mathbb{Z}^+$, the state $x \in \mathbb{R}^n$, control $u(t) \in \mathbb{R}^m$, disturbance $w \in \mathbb{R}^r$, and output $z \in \mathbb{R}^p$. It is assumed that the disturbance $w(t)$ is with finite l_2 norm. Denote the finite set $\mathfrak{I} = \{1, \cdots, N\}$, which stands for the collection of finite discrete modes. For any subsystem $i \in \mathfrak{I}$, the state matrices A_i, B_i, B_i^w, C_i, D_i and D_i^w are constant matrices of appropriate dimensions.

The problem being investigated here is to design not only the static state feedback gains K_i for each subsystem but also the switching signals, also in static state feedback form, i.e., $\sigma(x) : x \mapsto i$, such that the closed-loop switched system

$$\begin{cases} x(t+1) &= (A_{\sigma(x)} + B_{\sigma(x)} K_{\sigma(x)}) x(t) + B_{\sigma(x)}^w w(t) \\ z(t) &= (C_{\sigma(x)} + D_{\sigma(x)} K_{\sigma(x)}) x(t) + D_{\sigma(x)}^w w(t) \end{cases}$$

is exponentially stable with a bounded l_2 induced gain from w to z. To make the problem nontrivial, it is assumed that none of the subsystems (1) is stabilizable. In general, the design of continuous-variable control laws and switching signals are coupled together, and the co-design of K_i and $\sigma(x)$ as formulated above is a challenging task.

III. Switching Stabilization

This section aims to characterize switching signals in static state feedback form, i.e., $\sigma(x) : x \mapsto i$, such that the following autonomous switched system

$$x(t+1) = A_{\sigma(x)} x(t) \tag{2}$$

is exponentially stable to the origin. Notice that for all the subsystems in the form of (2), the origin is the common equilibrium.

To be precise, the exponential stability of the switched system (2) is defined as follows, e.g. [1]

Definition 1: The origin of the system (2) is exponentially stable if all trajectories satisfy

$$\|x(t)\| \leq \kappa \xi^t \|x_0\| \tag{3}$$

for some $\kappa > 0$ and $0 < \xi < 1$. Here $\|\cdot\|$ stands for standard Euclidian norm in $\mathbb{R}^n$. □

First, we recall a well-known approach in switched systems literature to guarantee exponentially stability using multiple Lyapunov functions.

A. Multiple Lyapunov Function Theorem

Since we assume that none of the subsystems is stabilizable, there does not exist a Lyapunov function for the subsystems, $x(t+1) = A_i x(t)$, in a classical sense. However, it is still possible to restrict ourselves in a certain region of the state space, say $\Omega_i \subset \mathbb{R}^n$, so that the abstracted energy of the i-th subsystem is decreasing along the trajectories inside this region (there is no requirement on the trajectories outside the region Ω_i). This idea is captured by the concept of Lyapunov-like function.

Definition 2 (Lyapunov-like function): By saying that a subsystem has an associated Lyapunov-like function $V_i : \mathbb{R}^n \to \mathbb{R}$ in a region Ω_i, we mean that

1) There exist constant scalars $\beta_i \geq \alpha_i > 0$ such that
$$\alpha_i \|x(t)\|^2 \leq V_i(x(t)) \leq \beta_i \|x(t)\|^2$$
hold for any $x(t) \in \Omega_i$;
2) For all $x(t) \in \Omega_i$ and $x(t) \neq 0$,
$$\Delta V_i(x(t)) = V_i(x(t+1)) - V_i(x(t)) < 0.$$

□

The first condition implies positiveness and radial unboundedness for $V_i(x)$ when $x \in \Omega_i$, while the second condition guarantees the decreasing of the value of $V_i(x)$ along trajectories of the i-th subsystem inside Ω_i. Notice that it is possible that $x(t) \in \Omega_i$ while $x(t+1) \notin \Omega_i$.

Suppose that the union of all these regions Ω_i cover the whole state space. Then we obtain a set of Lyapunov-like functions. To study the global stability of the switched systems, one needs to concatenate these Lyapunov-like functions together and form a non-traditional Lyapunov function, called multiple Lyapunov function (MLF). MLF is proved to be a powerful tool for studying the stability of switched systems, see e.g. [3], [16], [12], [6].

Theorem 1: Suppose that each subsystem has an associated Lyapunov-like function V_i in its active region Ω_i, and that $\bigcup_i \Omega_i = \mathbb{R}^n$. Let $\mathcal{S}$ be a class of switching sequences such that σ can take value i only if $x(t) \in \Omega_i$, and in addition

$$V_j(x(t_{i,j})) \leq V_i(x(t_{i,j})),$$

where $t_{i,j}$ denotes the time point that the switching from subsystem i to subsystem j occurs, i.e., $x(t_{i,j} - 1) \in \Omega_i$ while $x(t_{i,j}) \in \Omega_j$. Then, the switched linear system (2) is exponentially stable under the switching signals $\sigma \in \mathcal{S}$. □

B. Partition of the state space

In the above MLF theorem, it is critical to select Ω_i to divide the whole state space $\mathbb{R}^n$, so as to facilitate the identification of the Lyapunov-like functions $V_i(x)$ for each subsystem within a certain region. For this purpose, it is necessary to require that the union of all these regions Ω_i cover the whole state space, i.e., $\Omega_1 \bigcup \Omega_2 \cdots \bigcup \Omega_N = \mathbb{R}^n$, which is called the covering property.

Since we will restrict our attention to quadratic Lyapunov-like functions, we consider regions given (or approximated) by quadratic forms

$$\Omega_i = \{x \in \mathbb{R}^n : \ x^T Q_i x \geq 0\},$$

where $Q_i \in \mathbb{R}^{n\times n}$ are symmetric matrices, and $i \in \{1, \cdots, N\}$. The following lemma gives a sufficient condition for the covering property for regions given by quadratic forms [18].

Lemma 1: [18] If for every $x \in \mathbb{R}^n$

$$\sum_{i=1}^{N} \theta_i x^T Q_i x \geq 0, \tag{4}$$

where $\theta_i \geq 0$, $i \in \Im$, then $\bigcup_{i=1}^N \Omega_i = \mathbb{R}^n$. □

Consider the *largest region function strategy*, i.e.,

$$\sigma(x) = \arg\left(\max_{i\in\Im} x^T Q_i x\right). \tag{5}$$

This is due to the selection of subsystems (at state x) corresponding to the largest value of the region function $x^T Q_i x$. This switching strategy was previously introduced in [18] for continuous-time switched linear systems.

C. Quadratic Lyapunov-like Functions

In this subsection, we derive conditions expressed as matrix inequalities for the existence of a quadratic Lyapunov-like function, $V_i(x) = x^T P_i x$, assigned to each region Ω_i. By definition, the Lyapunov-like function $V_i(x) = x^T P_i x$ needs to satisfy the following two conditions:

1) Condition 1: There exist constant scalars $\beta_i \geq \alpha_i > 0$ such that

$$\alpha_i \|x(t)\|^2 \leq V_i(x(t)) \leq \beta_i \|x(t)\|^2$$

holds for any $x(t) \in \Omega_i$.

For a quadratic Lyapunov-like function candidate $V_i(x(t)) = x(t)^T P_i x(t)$, this means

$$\alpha_i x(t)^T I x(t) \leq x(t)^T P_i x(t) \leq \beta_i x(t)^T I x(t)$$

holds for $x(t)^T Q_i x(t) \geq 0$. That is

$$\begin{cases} x(t)^T(\alpha_i I - P_i)x(t) \leq 0 \\ x(t)^T(P_i - \beta_i I)x(t) \leq 0 \end{cases}$$

holds for $x(t)^T Q_i x(t) \geq 0$. Applying the $\mathcal{S}$-procedure [2], the above constrained inequalities follow from the LMIs

$$\begin{cases} \alpha_i I - P_i + \eta_i Q_i \leq 0 \\ P_i - \beta_i I + \rho_i Q_i \leq 0, \end{cases} \tag{6}$$

where $\eta_i \geq 0$ and $\rho_i \geq 0$ are unknown scalars. Define two scalars, $\alpha = \min_{i\in\Im}\{\alpha_i\}$ and $\beta = \max_{i\in\Im}\{\beta_i\}$. Notice that $0 < \alpha \leq \beta$. While normalizing $\beta = 1$ by resetting α as $\frac{\alpha}{\beta}$, η_i as $\frac{\eta_i}{\beta}$, and ρ_i as $\frac{\rho_i}{\beta}$, we obtain

$$\alpha I + \eta_i Q_i \leq P_i \leq I - \rho_i Q_i. \tag{7}$$

2) Condition 2: For all $x(t) \in \Omega_i$, $x(t) \neq 0$,

$$\Delta V_i(x(t)) = V_i(x(t+1)) - V_i(x(t)) < 0,$$

where $x(t+1) = A_i x(t)$.

This is equivalent to

$$x(t)^T[A_i^T P_i A_i - P_i]x(t) < 0 \tag{8}$$

for $x(t) \in \Omega_i$.

In order to transform the above constrained matrix inequality into equivalent unconstrained form, let's recall the Finsler's Lemma [2], which has been used previously in the control literature mainly for eliminating design variables in matrix inequalities.

Lemma 2 (Finsler's Lemma): Let $\zeta \in \mathbb{R}^n$, $P = P^T \in \mathbb{R}^{n\times n}$, and $H \in \mathbb{R}^{m\times n}$ such that $rank(H) = r < n$. The following statements are equivalent:

1) $\zeta^T P \zeta < 0$, for all $\zeta \neq 0$, $H\zeta = 0$;
2) $\exists X \in \mathbb{R}^{n\times m}$ such that $P + XH + H^T X^T < 0$. □

Applying the Finsler's Lemma to (8), with

$$P = \begin{bmatrix} -P_i & 0 \\ 0 & P_i \end{bmatrix}, \ \zeta = \begin{bmatrix} x(t) \\ x(t+1) \end{bmatrix},$$

$X = \begin{bmatrix} F_i \\ G_i \end{bmatrix}$, and $H = \begin{bmatrix} A_i & -I \end{bmatrix}$, then (8) is equivalent to

$$\zeta^T \begin{bmatrix} A_i^T F_i^T + F_i A_i - P_i & A_i^T G_i^T - F_i \\ G_i A_i - F_i^T & P_i - G_i - G_i^T \end{bmatrix} \zeta < 0$$

for $\zeta^T \begin{bmatrix} Q_i & 0 \\ 0 & 0 \end{bmatrix} \zeta \geq 0$. Here $F_i, G_i \in \mathbb{R}^{n\times n}$ are unknown matrices.

Applying the $\mathcal{S}$-procedure, the above constrained stability condition is implied by the following unconstrained condition for unknown matrices $P_i = P_i^T$, $Q_i = Q_i^T$, F_i, $G_i \in \mathbb{R}^{n\times n}$, and scalars $\mu_i \geq 0$,

$$\begin{bmatrix} A_i^T F_i^T + F_i A_i - P_i + \mu_i Q_i & A_i^T G_i^T - F_i \\ G_i A_i - F_i^T & P_i - G_i - G_i^T \end{bmatrix} < 0.$$

Combining *Condition 1* and *2*, we introduce methods to find quadratic Lyapunov-like functions for each subsystem within certain regions in the state space, which guarantee that the abstract energy of the subsystem is decreasing while staying within these regions. The next step is to properly patch these quadratic Lyapunov-like functions together, so as to obtain a global piecewise quadratic Lyapunov function to guarantee the decreasing of the abstract energy for the whole switched system. This is done in the next subsection based on the MLF theorem.

D. Switching Condition

Following Theorem 1, in order to guarantee exponential stability we also need to make sure that

1) Subsystem i is active only when $x(t) \in \Omega_i$,
2) When switching occurs, it is required to guarantee that the Lyapunov function value is not increasing.

To verify the first condition, suppose that the covering condition (4) holds, i.e., $\sum_{i=1}^N \theta_i x^T Q_i x \geq 0$ for some $\theta_i \geq 0$, $i \in \Im$. Then, based on the largest region function strategy, namely,

$$\sigma(x) = \arg\left(\max_{i\in\Im} x^T Q_i x\right),$$

the state x with current active mode i satisfies $x^T Q_i x \geq 0$. This implies $x \in \Omega_i$. So the first condition holds for the largest region function strategy.

Secondly, assume that a switching, $i \to j$, occurs at time instant t, i.e., $x(t) \in \Omega_j$ while $x(t-1) \in \Omega_i$ for $i \neq j \in \Im$, it is required that $V_j(x(t)) \leq V_i(x(t))$.

This means that

$$x(t)^T[P_j - P_i]x(t) \leq 0 \tag{9}$$

and $x(t-1) \in \Omega_i$, $x(t) = A_i x(t-1) \in \Omega_j$.

Because the above inequality is non-strict, the Finsler's Lemma can not be directly applied. However, it is possible to obtain a similar relation for the non-strict case. In fact,

$$\exists X : \quad P + XH + H^T X^T \leq 0$$

implies that $\zeta^T P \zeta \leq 0$, for all $\zeta \neq 0$, $H\zeta = 0$. This can be seen by left multiplying ζ^T and right multiplying ζ to $P + XH + H^T X^T \leq 0$ and using $H\zeta = 0$.

Therefore, with

$$P = \begin{bmatrix} 0 & 0 \\ 0 & P_j - P_i \end{bmatrix}, \ \zeta = \begin{bmatrix} x(t-1) \\ x(t) \end{bmatrix},$$

$X = \begin{bmatrix} F_{ij} \\ G_{ij} \end{bmatrix}$, and $H = \begin{bmatrix} A_i & -I \end{bmatrix}$, (9) is implied by

$$\zeta^T \begin{bmatrix} A_i^T F_{ij}^T + F_{ij} A_i & A_i^T G_{ij}^T - F_{ij} \\ G_{ij} A_i - F_{ij}^T & P_j - P_i - G_{ij} - G_{ij}^T \end{bmatrix} \zeta \leq 0$$

for $\zeta^T \begin{bmatrix} Q_i & 0 \\ 0 & Q_j \end{bmatrix} \zeta \geq 0$. Here F_{ij}, $G_{ij} \in \mathbb{R}^{n \times n}$ are unknown matrices.

Applying the $\mathcal{S}$-procedure, the above constrained stability condition is implied by the following: there exist unknown matrices $P_i = P_i^T$, $Q_i = Q_i^T$, F_{ij}, $G_{ij} \in \mathbb{R}^{n \times n}$, and scalars $\mu_{ij} \geq 0$, such that the matrix

$$\begin{bmatrix} A_i^T F_{ij}^T + F_{ij} A_i + \mu_{ij} Q_i & A_i^T G_{ij}^T - F_{ij} \\ G_{ij} A_i - F_{ij}^T & P_j - P_i - G_{ij} - G_{ij}^T + \mu_{ij} Q_j \end{bmatrix}$$

is negative semi-definite.

E. Synthesis Condition

In summary, the above discussion can be presented as the following sufficient condition for the discrete-time linear system (2) to be exponentially stabilized.

Theorem 2: If there exist matrices P_i ($P_i = P_i^T$), Q_i ($Q_i = Q_i^T$), F_i, G_i, F_{ij}, and scalars $\nu > 0$, $\alpha > 0$, $\eta_i \geq 0$, $\rho_i \geq 0$, $\mu_i \geq 0$, $\mu_{ij} \geq 0$, $\theta_i \geq 0$, solving the optimization problem (10) for all $i, j \in \{1, \cdots, N\}$, $i \neq j$, then the largest region function strategy implies that the origin of the switched linear system (2) is exponentially stable with decay rate $\xi = \sqrt{1-\nu}$. □

Some remarks are in order. First, the optimization problem above is a Bilinear Matrix Inequality (BMI) problem, due to the product of unknown scalars and matrices. BMI problems are NP-hard, and not computationally efficient. However, practical algorithms for optimization problems over BMIs exist and typically involve approximations, heuristics, branch-and-bound, or local search. As suggested in [18] for the continuous-time case, one possible way to compute the BMI problem is to grid up the unknown scalars, and then solve a set of LMIs for fixed values of these parameters. It is argued that the gridding of the unknown scalars can be made quite sparsely [18].

It can be shown that the introduction of multiplier matrices, like F_i, G_i etc., gives a lot of flexibility, and several known stability conditions in the literature can be reduced to a special selection of these multiplier matrices, see e.g. [7]. In addition, these multiplier matrices would make the co-design of continuous feedback controllers and switching laws trackable. This is explored in the next section.

IV. SWITCHED STATE FEEDBACK

This section focuses on the co-design of static state feedback gains K_i, and switching laws $\sigma(x)$ so that the closed-loop switched linear system

$$x(t+1) = (A_{\sigma(x)} + B_{\sigma(x)} K_{\sigma(x)}) x(t) \tag{11}$$

is exponentially stable to the origin. An important aspect of the matrix inequality conditions in Theorem 2 is that there is no cross product between two unknown matrices, which makes it possible to represent a sufficient condition for this co-design problem as follows.

Theorem 3: If there exist matrices P_i ($P_i = P_i^T$), Q_i ($Q_i = Q_i^T$), R_i, G_i, and scalars $\nu > 0$, $\alpha > 0$, $\eta_i \geq 0$, $\rho_i \geq 0$, $\mu_i \geq 0$, $\mu_{ij} \geq 0$, $\theta_i \geq 0$ that solve the optimization problem

$$\max \nu$$
$$s.t. \begin{cases} \alpha I + \eta_i Q_i \leq P_i \leq I - \rho_i Q_i \\ \begin{bmatrix} -P_i + \mu_i Q_i + \nu I & A_i^T G_i^T + R_i^T B_i^T \\ G_i A_i + B_i R_i & P_i - G_i - G_i^T \end{bmatrix} \leq 0, \\ \begin{bmatrix} \mu_{ij} Q_i & A_i^T G_i^T + R_i^T B_i^T \\ G_i A_i + B_i R_i & P_j - P_i - G_i - G_i^T + \mu_{ij} Q_j \end{bmatrix} \leq 0 \\ \theta_1 Q_1 + \cdots + \theta_N Q_N \geq 0 \end{cases}$$

for all $i, j \in \Im$, $i \neq j$, then the state feedback gains given by the solution of

$$G_i K_i = R_i, \quad i \in \Im \tag{12}$$

along with the switching strategy

$$\sigma(x) = \arg\max_{i \in \Im} x^T Q_i x \tag{13}$$

exponentially stabilize the switched system (11) with decay rate $\xi = \sqrt{1-\nu}$. □

Proof: The above conditions lead to

$$\begin{cases} \alpha_i I + \eta_i Q_i \leq P_i \leq \beta_i I - \rho_i Q_i \\ \begin{bmatrix} -P_i + \mu_i Q_i + \nu I & (A_i + B_i K_i)^T G_i^T \\ G_i (A_i + B_i K_i) & P_i - G_i - G_i^T \end{bmatrix} \leq 0, \\ \begin{bmatrix} \mu_{ij} Q_i & (A_i + B_i K_i)^T G_i^T \\ G_i (A_i + B_i K_i) & P_j - P_i - G_i - G_i^T + \mu_{ij} Q_j \end{bmatrix} \leq 0 \\ \theta_1 Q_1 + \cdots + \theta_N Q_N \geq 0 \end{cases}$$

which implies conditions in Theorem 2 with $F_i = 0$, $F_{ij} = 0$, and $G_{ij} = G_i$. Hence, the exponential stability of the switched control system (11) follows. ■

$$
\max \nu \quad s.t. \begin{cases} \alpha I + \eta_i Q_i \le P_i \le I - \rho_i Q_i \\ \begin{bmatrix} A_i^T F_i^T + F_i A_i - P_i + \mu_i Q_i + \nu I & A_i^T G_i^T - F_i \\ G_i A_i - F_i^T & P_i - G_i - G_i^T \end{bmatrix} \le 0, \\ \begin{bmatrix} A_i^T F_{ij}^T + F_{ij} A_i + \mu_{ij} Q_i & A_i^T G_{ij}^T - F_{ij} \\ G_{ij} A_i - F_{ij}^T & P_j - P_i - G_{ij} - G_{ij}^T + \mu_{ij} Q_j \end{bmatrix} \le 0 \\ \theta_1 Q_1 + \cdots + \theta_N Q_N \ge 0 \end{cases} \tag{10}
$$

V. Performance

Consider the discrete-time systems (1) with l_2-norm bounded disturbance w. The goal of this section is to guarantee that the l_2 induced gain from the disturbance w to the output z is below certain desirable bound.

A. MLF Theorem for Performance

To consider the l_2 gain performance, we first consider the switched autonomous systems

$$
\begin{cases} x(t+1) &= A_{\sigma(x)} x(t) + B^w_{\sigma(x)} w(t) \\ z(t) &= C_{\sigma(x)} x(t) + D^w_{\sigma(x)} w(t) \end{cases} \tag{14}
$$

and extend Theorem 1.

Proposition 1: Suppose each subsystem has an associated Lyapunov-like function V_i in its active region Ω_i with finite l_2 gain performance, each with equilibrium point $x = 0$. This means that

1) There exist constant scalars $\beta_i \ge \alpha_i > 0$ such that

$$
\alpha_i \|x(t)\|^2 \le V_i(x(t)) \le \beta_i \|x(t)\|^2
$$

hold for any $x(t) \in \Omega_i$;

2) For all $x(t) \in \Omega_i$ and $x(t) \ne 0$,

$$
\Delta V_i(x(t)) + z(t)^T z(t) - \gamma_i^2 w(t)^T w(t) < 0.
$$

Also, suppose that $\bigcup_i \Omega_i = \mathbb{R}^n$. Let $\mathcal{S}$ be a class of piecewise-constant switching sequences such that σ can take value i only if $x(t) \in \Omega_i$, and in addition

$$
V_j(x(t_{i,j})) \le V_i(x(t_{i,j}))
$$

where $t_{i,j}$ denotes the time point that the switching from subsystem i to subsystem j occurs, i.e., $x(t_{i,j} - 1) \in \Omega_i$ while $x(t_{i,j}) \in \Omega_j$. Then, the switched linear system (1) is exponentially stable under the switching signals $\sigma \in \mathcal{S}$, and its l_2 induced gain is less than γ, where $\gamma = \max_i \gamma_i$. □

B. Synthesis Condition for Performance

In a parallel development to Section III, we consider piecewise quadratic Lyapunov functions and derive corresponding matrix inequalities.

The condition that for all $x(t) \in \Omega_i$ and $x(t) \ne 0$,

$$
\Delta V_i(x(t)) + z(t)^T z(t) - \gamma_i^2 w(t)^T w(t) < 0,
$$

means that

$$
x^T(t)[A_i^T P_i A_i - P_i] x(t) + z(t)^T z(t) - \gamma_i^2 w(t)^T w(t) < 0,
$$

for $x(t) \in \Omega_i$, and $z(t) = C_i x(t) + D_i^w w(t)$, $x(t+1) = A_i x(t) + B_i^w w(t)$. This can be transformed into a matrix inequality based on the Finsler's Lemma, with

$$
P = \begin{bmatrix} -P_i & 0 & 0 & 0 \\ 0 & P_i & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}, \ \zeta = \begin{bmatrix} x(t) \\ x(t+1) \\ z(t) \\ w(t) \end{bmatrix},
$$

$$
X = \begin{bmatrix} F_{1i} & F_{2i} \\ G_{1i} & G_{2i} \\ H_{1i} & H_{2i} \\ J_{1i} & J_{2i} \end{bmatrix}, \ H = \begin{bmatrix} A_i & -I & 0 & B_i^w \\ C_i & 0 & -I & D_i^w \end{bmatrix}.
$$

Analogously, we can obtain the following sufficient conditions for the discrete-time switched linear system (1) to be stabilized exponentially with l_2 gain less than γ.

Theorem 4: If there exist matrices P_i ($P_i = P_i^T$), Q_i ($Q_i = Q_i^T$), F_{1i}, G_{1i}, H_{1i}, J_{1i}, F_{2i}, G_{2i}, H_{2i}, J_{2i}, F_{ij}, G_{ij}, and scalars $\alpha > 0$, $\eta_i \ge 0$, $\rho_i \ge 0$, $\gamma > 0$, $\mu_i \ge 0$, $\mu_{ij} \ge 0$, $\theta_i \ge 0$ that solve the optimization problem (17), then the switched system (14) can be exponentially stabilized with l_2 gain less than γ by the largest region function strategy. □

Next, consider the following switched control system (1) and the aim is to find switching signal $\sigma(x)$ and static state feedback gains K_i, such that the closed-loop switched system is exponentially stable with finite l_2 induced gain. A sufficient condition can be expressed in the following theorem.

Theorem 5: If there exist matrices P_i ($P_i = P_i^T$), Q_i ($Q_i = Q_i^T$), F_{1i}, G_{1i}, H_{1i}, J_{1i}, F_{2i}, G_{2i}, H_{2i}, J_{2i}, F_{ij}, G_{ij}, and scalars $\alpha_i > 0$, $\beta_i > 0$, $\eta_i \ge 0$, $\rho_i \ge 0$, $\gamma > 0$, $\mu_i \ge 0$, $\mu_{ij} \ge 0$, $\theta_i \ge 0$ that solve the optimization problem

$$
\min \gamma \quad \begin{cases} \alpha_i I + \eta_i Q_i \le P_i \le \beta_i I - \rho_i Q_i \\ \Lambda_i + U_i + U_i^T < 0 \\ \begin{bmatrix} \mu_{ij} Q_i & A_i^T G_{1i}^T + R_i^T B_i^T \\ G_{1i} A_i + B_i R_i & P_j - P_i - G_{1i} - G_{1i}^T + \mu_{ij} Q_j \end{bmatrix} \le 0 \\ \theta_1 Q_1 + \cdots + \theta_N Q_N \ge 0 \end{cases}
$$

where

$$
\Lambda_i = \begin{bmatrix} -P_i + \mu_i Q_i & 0 & 0 & 0 \\ 0 & P_i & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix},
$$

$$
\begin{aligned}
&\min \ \gamma \\
&s.t. \begin{cases}
\alpha I + \eta_i Q_i \le P_i \le I - \rho_i Q_i \\
\Lambda_i + U_i + U_i^T < 0 \\
\begin{bmatrix} A_i^T F_{ij}^T + F_{ij} A_i + \mu_{ij} Q_i & A_i^T G_{ij}^T - F_{ij} \\ G_{ij} A_i - F_{ij}^T & P_j - P_i - G_{ij} - G_{ij}^T + \mu_{ij} Q_j \end{bmatrix} \le 0 \\
\theta_1 Q_1 + \cdots + \theta_N Q_N \ge 0
\end{cases}
\end{aligned}
\tag{17}
$$

where

$$
\Lambda_i = \begin{bmatrix} -P_i + \mu_i Q_i & 0 & 0 & 0 \\ 0 & P_i & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}, \quad U_i = \begin{bmatrix} F_{1i} A_i + F_{2i} C_i & -F_{1i} & -F_{2i} & F_{1i} B_i^w + F_{2i} D_i^w \\ G_{1i} A_i + G_{2i} C_i & -G_{1i} & -G_{2i} & G_{1i} B_i^w + G_{2i} D_i^w \\ H_{1i} A_i + H_{2i} C_i & -H_{1i} & -H_{2i} & H_{1i} B_i^w + H_{2i} D_i^w \\ J_{1i} A_i + J_{2i} C_i & -J_{1i} & -J_{2i} & J_{1i} B_i^w + J_{2i} D_i^w \end{bmatrix},
$$

for all $i, j \in \{1, \cdots, N\}$ $i \neq j$.

$$
U_i = \begin{bmatrix} B_i^w F_{2i} & A_i G_{1i} + B_i R_i + B_i^w G_{2i} & B_i^w H_{2i} & B_i^w J_{2i} \\ 0 & -G_{1i} & 0 & 0 \\ -F_{2i} & -G_{2i} & -H_{2i} & -J_{2i} \\ D_i^w F_{2i} & C_i G_{1i} + D_i R_i + D_i^w G_{2i} & D_i^w H_{2i} & D_i^w J_{2i} \end{bmatrix}
$$

for all $i, j \in \{1, \cdots, N\}$ $i \neq j$, then the linear system (1) can be exponentially stabilized with l_2 gain less than γ by the state feedback gains

$$
G_{1i} K_i = R_i, \quad i \in \Im
$$

along with the largest region function strategy. □

To show this, use a transposed version of Theorem 4 with $F_{1i} = 0$, $H_{1i} = 0$ and $J_{1i} = 0$, $G_{ij} = G_{1i}$ and $R_i = K_i G_{1i}$.

VI. Concluding Remarks

In this paper, the co-design of continuous-variable controllers and discrete-event switching logics, both in the state feedback form, is constructively shown for a class of discrete-time switched linear systems. The exponential stability and l_2 induced gain performance are investigated based on multiple quadratic Lyapunov-like functions. Sufficient synthesis conditions are proposed as an optimization problem with bilinear matrix inequality constraints.

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