P-adic Analysis and the Igusa local Zeta function

Introduction

These lectures are an introduction to p-adic numbers, p-adic analysis and the Igusa local zeta function. P-adic analysis was introduced by Kurt Hensel in the very late 1800's and early 1900's and Hensel's student Helmut Hasse did much in the early 1900s to establish the theory.

The idea is that is you start with the rational numbers and change the way you measure the distance from zero of a rational number (or, equivalently, change the absolute value on the rationals) so that a number is really close to zero if it is divisible by a lot of powers of p. Thus, divisibility by higher and higher powers of p can be thought of as a kind of convergence.

To get a full set of calculus tools in this theory, you consider the *completion* of the rational numbers with respect to this new absolute value. This new completed field is analogous to the real numbers and is called the field of p-adic numbers, denoted by $\mathbb{Q}_p$. Just as the rational numbers sit inside the real numbers, so too do the rational numbers sit inside the p-adics, for each p. Together the real numbers and the p-adic numbers for each p provide all the possible different ways to arrange the rational numbers around zero so that calculus makes sense. The fields are together all the different possible "shadows" of the rational numbers,

The p-adic numbers share some properties with the real numbers but there are also some big differences. One big difference is that the real numbers have only themselves and the complex numbers as extension fields while the p-adic numbers have infinitely many distinct extension fields. Studying p-adic analysis, as you will see, means using a very unusual combination of algebra and analysis. Although p-adic analysis was developed as a tool in number theory (to extend calculus-type thinking to the very algebraic, modular arthimetic way that number theorists were thinking), the p-adics have been useful in many other areas of mathematics (in particular in group theory, representation theory, and algebraic geometry) and in physics.

Lecture 1 will introduce the p-adic numbers and p-adic integration and the Igusa local zeta function for simple cases that can be computed directly.
Lecture 2 will introduce the Poincaré series as a generating function associated with the Igusa Local Zeta function. This lecture will discuss

Igusa's stationary phase formula for computing the zeta function and its Poincaré series.

Lecture 3 will introduce the resolution of singularities for curves as a way to illustrate Igusa's original proof of the rationality of his local zeta function. We will also explain the Bernstein polynomial and some conjectures that connect it to the local zeta function. Finally I hope to say some things about a theorem of Denef and Meuser that uses the Weil zeta function to say something about the Igusa local zeta function.

This is beautiful mathematics and I hope these lectures will highlight the richness of the mathematical ideas.

Technology: I will introduce PARI/GP.

Home page for PARI/GP: http://pari.math.u-bordeaux.fr

Bibliography:

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