Example to explain how to use Hironaka's resolution of singularities to compute the Igusa local zeta function for a curve

Hironaka's resolution of singularities states that for any polynomial $f(x_1, x_2, \cdots, x_n)$ in n variables with coefficients in a field $\mathbb{K}$ of characteristic 0 there exists a manifold Y and a projection map (morphism) $h : Y \to \mathbb{K}$ such that

(1) Each irreducible component of $foh = 0$ is nonsingular
(2) The components of $foh = 0$ in Y intersect transversally.

It is always possible to find h such that

$$Z(s) = \int_{\mathbb{Z}_p^n} |f(x)|^s \; dx = \int_{h^{-1}(\mathbb{Z}_p^n)} |foh(y)|^s \; h^*(dx)$$

where h^* is the change in measure and $h^{-1}(\mathbb{Z}_p^n)$ is chosen carefully.

The key is that $h^{-1}(\mathbb{Z}_p^n)$ will be a union of a finite number of non intersecting neighborhoods D_i in Y in which

$$\begin{aligned} foh(y) &= u(y) y_1^{N_1} y_2^{N_2} \cdots y_n^{N_n} \\ h^*(dx) &= v(y) y_1^{m_1 - 1} y_2^{m_2 - 1} \cdots y_n^{m_n - 1} \; dy \end{aligned}$$

and $u(y)$ and $v(y)$ either do not intersect the axes of D_i or they have transverse intersections. Thus the zeta function becomes a finite sum of integrals of essentially the form

$$\int_{D_i} |y_1|^{N_1 s + m_1 - 1} |y_2|^{N_2 s + m_2 - 1} \cdots |y_n|^{N_n s + m_n - 1} \; dy$$

and these integral can all be computed.

Suppose that $(0, 0)$ is a singular point on the curve $f(x, y) = 0$. (We can also blow up a point (a, b) that is singular on $f(x, y) = 0$ but here we will focus on blowing up (0,0).) The problem is that since

$$Z(s) = \int_{\mathbb{Z}_p^n} |f(x)|^s \; dx = \int_{\mathbb{Z}_p^n \setminus (0,0)} |f(x)|^s \; dx$$

and the set $\mathbb{Z}_p^n \setminus (0, 0)$ is not compact the integral could end up being a sum of integrals over an infinite number of neighborhoods and in that case it might very well not be rational.

We will construct a quasi projective variety $Y = \mathbb{A}ff^2(\mathbb{Z}_p) \times \mathbb{P}^1(\mathbb{Z}_p)$ and take a nonsingular closed subset $V(\mathbb{Z}_p) \subset Y$ such that $V = \{(x, y : u, v) | \; vx - uy = 0\}$. Remember that in projective space $[u, v] \neq [0, 0]$ and $[u, v] \equiv [tu, tv]$.

Note that this set up implies that if $x \neq 0$ then $\frac{y}{x} = \frac{v}{u}$ so $[u,v] = [1,\frac{v}{u}]u = [1,\frac{y}{x}]$ so in Y the point $(x,y:u,v) = (x,y:1,\frac{y}{x})$. Similarly if $y \neq 0$ then $\frac{x}{y} = \frac{u}{v}$ so $[u,v] = [\frac{u}{v},1]v = [\frac{x}{y},1]$ and so the point $(x,y:u,v) = (x,y:\frac{x}{y},1)$. If both x and y are not zero then the point $(x,y:u,v) = (x,y:1,\frac{y}{x}) = (x,y:\frac{x}{y},1)$.

Thus for example, $(1,2:1/2,1) = (1,2:1,2)$ but the point $(2,0:1,0)$ has only one representation.

Now there is a projection $pr : V \to \mathbb{Z}_p^2$ such that $pr(x,y:u,v) = (x,y)$ and $pr^{-1}(x,y) = (x,y,1,\frac{y}{x})$ if $x \neq 0$ or $(x,y:\frac{x}{y}),1)$ if $y \neq 0$ and these points are the same if neither x nor y is 0. However $pr^{-1}(0,0) = (0,0) \times \mathbb{P}^1(\mathbb{Z}_p)$ so $pr^{-1}(0,0)$ is not well-defined. It is true however that $pr : V \setminus pr^{-1}(00) \to \mathbb{Z}_p^2 \setminus (0,0)$ is an isomorphism.

With this set up we see that $V = W \cup W'$. We have that $W = \{(x,y:u,v)|x \neq 0\} = \{x,y:1,\frac{y}{x})\}$. We can take coordinates $x_1 = x$ and $y_1 = \frac{y}{x}$ for our axes in W. Now $W' = \{(x,y:u,v)|y \neq 0\} = \{x,y:\frac{x}{y})\}$. We can take coordinates $\xi_1 = y$ and $\eta_1 = \frac{x}{y}$ for our axes in W'.

0.1. **Computation of the Igusa local zeta for** $f(x,y) = x^3 + y^2$ **using resolution of singularities.** Resolve $f(x,y) = x^3 + y^2$ to find the Igusa local zeta function.

$$Z(t) = \int_{\mathbb{Z}_p \times \mathbb{Z}_p} |f(x,y)|^s \, dx \, dy \; = \sum_{i=1}^{4} \int_{D_i} |f \circ h|^s |h^*(dx \, dy)|$$

Where h^* is the change in measure and $h^{-1}(\mathbb{Z}_p \times \mathbb{Z}_p) = D_1 \cup D_2 \cup D_3 \cup D_4$.

Step 1:
Let $h : W \to \mathbb{Z}_p^2$ such that $(x_1,y_1) = (x,\frac{y}{x})$ and $h : W' \to \mathbb{Z}_p^2$ such that $(\xi_1,\eta_1) = (y,\frac{x}{y})$. Therefore,

$$(x,y) = \begin{cases} (x_1, x_1 y_1) & \text{in } W \\ (\xi_1 \eta_1, \xi_1) & \text{in } W' \end{cases}$$

So,

$$f(x,y) = \begin{cases} x_1^2(x_1 + y_1^2) \text{ in } W \\ \xi_1^2(1 + \xi_1 \eta_1^3) \text{ in } W' \end{cases}$$

Now looking at f we see that in W' the curve $f = 0$ is resolved because $1 + \xi_1 \eta_1^3$ does not intersect the ξ_1 or η_1 axes. However, $f = 0$ in W is still singular at $(x_1,y_1) = (0,0)$ so we must blow up the point $(0,0)$ in W.

Since x and y are in $\mathbb{Z}_p$, $|x|_p \leq 1$, $|y|_p \leq 1$, so either $|\frac{x}{y}|_p \leq 1$ or $|\frac{y}{x}|_p < 1$. Therefore, we can assume that $|x_1| \leq 1, |y_1| \leq 1$ and $|\xi_1| \leq 1, |\eta_1| < 1$. This means that $(x_1,y_1) \in \mathbb{Z}_p^2$ and $(\xi_1,\eta_1) \in \mathbb{Z}_p \times p\mathbb{Z}_p = D_1$.

Step 2:
Now we resolve $f(x,y) = x_1^2(x_1 + y_1^2)$ in W.

$$(x_2, y_2) = (y_1, \frac{x_1}{y_1})$$

$$(\xi_2, \eta_2) = (x_1, \frac{y_1}{x_1})$$

Therefore,

$$(x_1, y_1) = \begin{cases} (x_2y_2, x_2) \text{in } W_1 \\ (\xi_2, \xi_2\eta_2) \text{in } W_1' \end{cases}$$

So,

$$f(x,y) = \begin{cases} x_2^3y_2^2(x_2 + y_2) \text{in } W_1 \\ \xi_2^3(1 + \xi_2\eta_2^2) \text{in } W_1' \end{cases}$$

Now looking at f we see that in W_1' the curve $f = 0$ is resolved because $1 + \xi_2\eta_2^2$ does not intersect the ξ_2 or η_2 axes. However, the $f = 0$ in W_1 is still singular at $(x_2, y_2) = (0,0)$ so we must blow up the point $(0,0)$ in W_1.

Since x_1 and y_1 are in $\mathbb{Z}_p$, $|x_1|_p \leq 1$, $|y_1|_p \leq 1$, so either $|\frac{x_1}{y_1}|_p \leq 1$ or $|\frac{y_1}{x_1}|_p < 1$. Therefore, we can assume that $|x_2| \leq 1, |y_2| \leq 1$ and $|\xi_2| \leq 1, |\eta_2| < 1$. This means that $(x_2, y_2) \in \mathbb{Z}_p^2$ and $(\xi_2, \eta_2) \in \mathbb{Z}_p \times p\mathbb{Z}_p = D_2$.

Step 3:
Now, we resolve $f(x,y) = x_2^3y_2^2(x_2 + y_2)$.

$$(x_3, y_3) = (x_2, \frac{y_2}{x_2})$$

$$(\xi_3, \eta_3) = (y_2, \frac{x_2}{y_2})$$

Therefore,

$$(x_2, y_2) = \begin{cases} (x_3, x_3y_3) \text{in } W_2 \\ (\xi_3\eta_3, \xi_3) \text{in } W_2' \end{cases}$$

Therefore,

$$f(x,y) = \begin{cases} x_3^6y_3^2(1 + y_3) \text{in } W_2 \\ \xi_3^6\eta_3^3(1 + \eta_3) \text{in } W_2' \end{cases}$$

Both of these curves are non singular with transverse intersections. Now we must find D_3 and D_4. Since x_2 and y_2 are in $\mathbb{Z}_p$, $|x_2|_p \leq 1$, $|y_2|_p \leq 1$, so either $|\frac{x_2}{y_2}|_p \leq 1$ or $|\frac{y_2}{x_2}|_p < 1$. Therefore $|x_3| \leq 1, |y_3| \leq 1$ and $|\xi_3| \leq 1, |\eta_3| < 1$. This means that $(x_3, y_3) \in \mathbb{Z}_p^2 = D_4$ and $(\xi_3, \eta_3) \in \mathbb{Z}_p \times p\mathbb{Z}_p = D_3$. Therefore, $D_1 = D_2 = D_3 = \mathbb{Z}_p \times p\mathbb{Z}_p$ while $D_4 = \mathbb{Z}_p^2$.

Next we calculate the change in measure due to the change of variables in D_1. Here I am using differential form but I could also us the Jacobian to calculate the change in measure.

$$\begin{aligned} dx \wedge dy &= d(\xi_1, \eta_1) \wedge d\xi_1 \\ &= \xi_1 \ d\eta_1 \wedge d\xi_1 \\ &= -\xi_1 \ d\xi_1 \wedge d\eta_1 \end{aligned}$$

Now, we calculate the first partial integral for the zeta function over D_1.

$$\begin{aligned} Z_1(t) &= \int_{\mathbb{Z}_p \times p\mathbb{Z}_p} |\xi_1^2(1+\xi_1\eta_1^3)|^s \ |\xi_1| \ d\xi_1 d\eta_1 \\ &= \int_{\mathbb{Z}_p \times p\mathbb{Z}_p} |\xi_1|^{2s+1} \ d\xi_1 d\eta_1 \\ &= p^{-1} \int_{\mathbb{Z}_p} |\xi_1|^{2s+1} \ d\xi_1 \\ &= p^{-1} \frac{1-p^{-1}}{1-p^{-2}t^2} \end{aligned}$$

Next we calculate the change in measure due to the change of variables in D_2.

$$\begin{aligned} dx \wedge dy &= dx_1 \wedge d(x_1 y_1) \\ &= x_1 \ dx_1 \wedge dy_1 \\ dx \wedge dy &= \xi_2 \ d\xi_2 \wedge d(\xi_2\eta_2) \text{ or } x_2 y_2 \ d(x_2 y_2) \wedge dx_2 \\ &= \xi_2^2 \ d\xi_2 \wedge d\eta_2 \qquad\qquad - x_2^2 y_2 \ dx_2 \wedge dy_2 \end{aligned}$$

Now, calculate the second partial integral in the zeta function over D_2.

$$\begin{aligned} Z_2(t) &= \int_{\mathbb{Z}_p \times p\mathbb{Z}_p} |\xi_2^3(1+\xi_2\eta_2^2)|^s \ |\xi_2|^2 \ d\xi_2 d\eta_2 \\ &= \int_{\mathbb{Z}_p \times p\mathbb{Z}_p} |\xi_2|^{3s+2} \ d\xi_2 d\eta_2 \\ &= p^{-1} \frac{1-p^{-1}}{1-p{-}3t^3} \end{aligned}$$

Next we calculate the change in measure due to the change of variables in D_3 and in D_4.

$$\begin{aligned} dx \wedge dy &= -(\xi_3\eta_3)^2 \xi_3 \ d(\xi_3\eta_3) \wedge d\xi_3 \qquad - x_3^2 x_3 y_3 \ dx_3 \wedge d(x_3 y_3) \\ &= \xi_3^4 \eta_3^2 \ d\xi_3 \wedge d\eta_3 \qquad\quad - x_3^4 y_3 \ dx_3 \wedge dy_3 \end{aligned}$$

Now, calculate the third and fourth partial integral of the zeta function over D_3 and D_4.

$$\begin{aligned} Z_3(t) &= \int_{\mathbb{Z}_p \times p\mathbb{Z}_p} |\xi_3^6 \eta_3^3 (1+\eta_3)|^s \ |\xi_3|^4 |\eta_3|^2 \ d\xi_3 d\eta_3 \\ &= \int_{\mathbb{Z}_p} |\xi_3|^{6s+4} \ d\xi_3 \quad \int_{p\mathbb{Z}_p} |\eta_3|^{3s+2} \ d\eta_3 \\ &= p^{-3} t^3 \left(\frac{1-p^{-1}}{1-p^{-5}t^6} \right) \left(\frac{1-p^{-1}}{1-p^{-3}t^3} \right) \end{aligned}$$

$$\begin{aligned} Z_4(t) &= \int_{\mathbb{Z}_p^2} |x_3^6 y_3^2 (1+y_3)|^s \ |x_3|^4 |y_3| \ dx_3 dy_3 \\ &= \int_{\mathbb{Z}_p} |x_3|^{6s+4} \ dx_3 \quad \int_{\mathbb{Z}_p} |y_3|^{2s+1} |y_3+1|^s \ dy_3 \\ &= \frac{1-p^{-1}}{1-p^{-5}t^6} \sum_{a mod p} \int_{a+p\mathbb{Z}_p} |y_3|^{2s+1} |y_3+1|^s \ dy_3 \\ &= \left(\frac{1-p^{-1}}{1-p^{-5}t^6} \right) \left((p-2)p^{-1} \ + \ \int_{p\mathbb{Z}_p} |y_3|^{2s+1} \ dy_3 \ + \ \int_{-1+p\mathbb{Z}_p} |y_3+1|^s \ dy_3 \right) \\ &= \frac{1-p^{-1}}{1-p^{-5}t^6} \left((p-2)p^{-1} \ + \ p^{-2}t^2 \frac{1-p^{-1}}{1-p^{-2}t^2} \ + \ p^{-1}t \frac{1-p^{-1}}{1-p^{-1}t} \right) \end{aligned}$$

The Igusa local zeta function, $Z(t) = Z_1(t) + Z_2(t) + Z_3(t) + Z_4(t)$, or

$$Z(t) = \frac{(1-p^{-1})(1-p^{-2}t+p^{-2}t^2-p^{-5}t^5)}{(1-p^{-1}t)(1-p^{-5}t^6)}$$