TROPICAL GEOMETRY PROBLEMS, DAY 2

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(1) Let K be any field with valuation. Prove that if a and b are elements of K with $v(a) \neq v(b)$, then $v(a+b) = \min\{v(a), v(b)\}$.

(2) In this example, we work with $K = \mathbb{Q}_5$, the field of 5-adic numbers. What is the tropicalization of $f = x - y + 25$? Prove that for any point (a, b) in the tropicalization such that a and b are integers, then there exists $x, y \in \mathbb{Q}_5$ with $f(x, y) = 0$, $v(x) = a$, and $v(y) = b$. (This is stronger than the fundamental theorem because we're showing solutions in $\mathbb{Q}_5$, not its algebraic closure.)

(3) Let K and f be as in the previous problem. If a and b are rational numbers such that (a, b) is in the tropical curve of f, then can you find x and y in $\overline{\mathbb{Q}}_5$, the algebraic closure of $\mathbb{Q}_5$, such that $f(x, y) = 0$, $v(x) = a$, and $v(y) = b$?

(4) Let K be $\mathbb{C}\{\{\pi\}\}$ and let I be the ideal in $K[x^{\pm}, y^{\pm}]$ generated by $f_1 = x - \pi$ and $f_2 = y - 4$. What is $V(I)$? What is $\mathrm{Trop}(I)$? Give a tropical basis for I and give a generating set for I which is not a tropical basis.

(5) Recall the following example from lecture: $K = \overline{\mathbb{Q}}_3$ and I has a tropical basis of $xy + y - x + 3$ and $z^{-1} + 2 - 3x$. We verified that the multiplicity at $(2, 1, 0)$ was 1. Use the balancing condition to show that all multiplicities are 1.

(6) Let $K = \mathbb{Q}_3$ and let $f = 3x^2 + y^2 + 1$. Verify that the point $(0, 0)$ is in the tropical hypersurface of f. Show, however, that there is no point $(x, y) \in \mathbb{Q}_3^2$ such that $f(x, y) = 0$ and $v(x) = v(y) = 0$. (Hint: consider the reduction of such a solution modulo 3.) Give an example of such (x, y) if they're instead allowed to be in $\overline{\mathbb{Q}}_3$.

(7) Prove that tropical hypersurfaces are connected through codimension 1.

(8) Understand the following definition: Let I be an ideal in the ring of Laurent polynomials $K[x_1^{\pm}, \ldots, x_n^{\pm}]$ and $w = (w_1, \ldots, w_n)$ be any point. Now let $t_1, \ldots, t_n$ be elements of K with $v(t_i) = w_i$ (you may need to enlarge your field to do this). Let R denote the subring of K of elements with non-negative valuation and $\mathfrak{m}$ the ideal in R of elements with positive valuation. Define $\mathrm{in}_w(I)$ to be the image of the ideal

$$(\{f(t_1x_1, \ldots, t_nx_n) \mid f(x_1, \ldots, x_n) \in I\} \cap R[x_1^{\pm}, \ldots, x_n^{\pm}]$$

in the ring $k[x_1^{\pm}, \ldots, x_n^{\pm}]$, where $k = R/\mathfrak{m}$.

(9) Let I be as in the previous problem. Prove that the tropical variety of I consists of those points $w \in \mathbb{R}^n$ such that $\mathrm{in}_w(I)$ does not contain 1.