1. Preliminaries: uniform convergence

Recall the following definition.

Definition 1.1. *A sequence* $(x_n)_{n\in\mathbf{N}} \subset \mathbf{R}$ *is said to* converge to $L \in \mathbf{R}$ *if for each* $\epsilon > 0$*, there exists* $N \in \mathbf{N}$ *such that* $|x_n - L| < \epsilon$ *whenever* $n \geq N$*.*

This definition is really the starting point for all of Calculus. It invests phrases like 'as Δx tends to zero' with a precise meaning and therefore allows us to speak about things like continuity, differentiation, and integration in a rigorously logical and not just intuitive fashion. A fundamental property of real numbers is that of *completeness*. It can be formulated in a number of ways (e.g. every set of real numbers that is bounded above has a least upper bound), but the best (arguably) uses the following additional notion.

Definition 1.2. *A sequence* $(x_n)_{n\in\mathbf{N}} \subset \mathbf{R}$ *is said to be* Cauchy *if for each* $\epsilon > 0$*, there exists* $N \in \mathbf{N}$ *such that* $|x_n - x_m| < \epsilon$ *whenever* $n, m \geq N$*.*

Observe that a convergent sequence is necessarily Cauchy: if (x_n) converges to L and $\epsilon > 0$ is given, then we can choose $N \in \mathbf{N}$ such that $|x_n - L| < \epsilon/2$ whenever $n \geq N$. Therefore if two indices $n, m \in \mathbf{N}$ are *both* larger than N, we get

$$|x_n - x_m| = |(x_n - L) - (x_m - L)| \leq |x_n - L| + |x_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence (x_n) is Cauchy. The completeness property of $\mathbf{R}$ is the converse assertion:

Completeness Axiom. *Every Cauchy sequence of real numbers converges.*

This property of $\mathbf{R}$ gives us a way of showing sequences converge without actually knowing anything about the limiting.

Here we will need a notion of convergence for sequences of *functions* rather than real numbers. To this end, let us fix a subset $S \subset \mathbf{R}$. For any $f : S \to \mathbf{R}$, we define

$$\|f\| = \|f\|_S := \sup_{x\in S} |f(x)|.$$

Of course it can happen that $\|f\| = \infty$. Consider for instance $S = (0, \infty)$, $f(x) = 1/x$. To keep things more manageable, we restrict attention to the set

$$\mathcal{B}(S) = \{f : S \to \mathbf{R}, \ \|f\| < \infty\}$$

of *bounded* real-valued functions on S. Note that $\mathcal{B}(S)$ is a vector space over $\mathbf{R}$. We leave the reader to verify

Proposition 1.3. $\|\cdot\|_S$ *is a* norm *on* $\mathcal{B}(S)$*. That is, for every* $f, g \in \mathcal{B}(S)$ *and every* $c \in \mathbf{R}$*, we have*

- *(positivity)* $\|f\| \geq 0$ *with equality if and only if* $f(x) = 0$ *for every* $x \in S$*.*
- *(homogeneity)* $\|cf\| = |c| \, \|f\|$*.*
- *(triangle inequality)* $\|f + g\| \leq \|f\| + \|g\|$ *.*

In other words, $\|\cdot\|_S$ has essentially the same properties on $\mathcal{B}(S)$ that the absolute value function has on $\mathbf{R}$. In particular, it gives a way to measure the distance between functions on S. That is, given $f, g \in \mathcal{B}(S)$, we can declare $\mathrm{dist}(f, g) = \|f - g\|$ to be the distance between f and g. And whenever you have a notion of distance, there is a corresponding notion of convergence.

Definition 1.4. *A sequence* $(f_n)_{n\in\mathbf{N}} \in \mathcal{B}(S)$ *is said to* converge uniformly *to a function* $g \in \mathcal{B}(S)$ *on* S *if for every* $\epsilon > 0$*, there exists* $N \in \mathbf{N}$ *such that* $\|f_n - g\| < \epsilon$ *whenever* $n \geq N$*.*

In other words, (f_n) converges uniformly to f if $\lim_{n\to\infty} \|f_n - f\| = 0$. The qualifier 'uniformly' is used here because there are other notions of convergence for sequences of functions, useful in other contexts, and they are fundamentally different than the one specified here. One can also rewrite the definition of Cauchy sequence of real numbers to come up with a definition of 'uniformly Cauchy' sequence of bounded functions. We leave doing this to the reader.

Let us remark before continuing that if (f_n) converges uniformly to f on S, then we have the *pointwise convergence*

$$\lim_{n\to\infty} f_n(t) = f(t) \text{ for every } t \in S.$$

The converse, however, is not true. Fairly straightforward counterexamples show that one can have $f_n(t) \to f(t)$ for every individual $t \in S$ yet still not get that $f_n \to f$ uniformly on S.

There are three fundamental facts about uniform convergence. The first is that bounded functions are complete with respect to uniform convergence.

Theorem 1.5. *A uniformly Cauchy sequence* $(f_n) \subset \mathcal{B}(S)$ *of functions is uniformly convergent.*

Proof. We first produce a candidate for the limit function $g : S \to \mathbf{R}$. Let $\epsilon > 0$ be given, and use the fact that (f_n) is Cauchy to obtain $N \in \mathbf{N}$ such that $n, m \geq N$ implies $\|f_n - f_m\| < \epsilon$. Then for any fixed $x \in S$, we have as a consequence that

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\| < \epsilon$$

whenever $n, m \geq N$. That is, $(f_n(x))_{n\in\mathbf{N}}$ is a Cauchy sequence of *real numbers*. It follows then from completeness of $\mathbf{R}$ that $(f_n(x))$ converges. We let $g(x) = \lim_{n\to\mathbf{N}} f_n(x)$. Since we can do this for each $x \in S$, we obtain a function $g : S \to \mathbf{R}$.

Next we show that g is bounded (i.e. $g \in \mathcal{B}(S)$). Taking $\epsilon = 1$, we again use the fact that (f_n) is uniformly Cauchy to obtain $N \in \mathbf{N}$ such that $n, m \geq N$ implies $\|f_n - f_m\| < 1$. In particular, $n \geq N$ implies $\|f_N - f_n\| < 1$. Thus if $M = \|f_N\| < \infty$, we get

$$\|f_n\| = \|(f_n - f_N) + f_N\| \leq \|f_n - f_N\| + \|f_N\| < 1 + M$$

for all $n \geq N$. Thus, for any $x \in S$, we have

$$|g(x)| = |\lim f_n(x)| = \lim |f_n(x)| \leq 1 + M,$$

because $|f_n(x)| \leq \|f_n\| < 1 + M$ for all $n \geq N$. Thus $\|g\| \leq 1 + M < \infty$, which proves that g is bounded.

Finally, we argue that f_n converges uniformly to g. Let $\epsilon > 0$ be given. Applying the fact that (f_n) is uniformly Cauchy one last time, we take $N \in \mathbf{N}$ such that $n, m \geq N$ implies $\|f_n - f_m\| < \epsilon/2$. Now if $x \in S$ is any particular point, we have already shown that $\lim_{n\to\infty} f_n(x) = g(x)$. So we can also find $N' \in \mathbf{N}$ such that $|f_n(x) - g(x)| < \epsilon/2$ whenever $n \geq N'$. Therefore, if $n \geq N$ and $m \geq \max\{N, N'\}$, we estimate

$$|g(x)-f_n(x)| = |(g(x)-f_m(x))+(f_n(x)-f_m(x))| \leq |g(x)-f_m(x)|+|f_n(x)-f_m(x)| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

In fact, since N (as opposed to N') was chosen independent of x, we get

$$\|f_n - g\| = \sup_{x \in S} |g(x) - f_n(x)| < \epsilon.$$

whenever $n \geq N$. This proves that (f_n) converges uniformly to g. □

The second fundamental fact about uniform convergence is that it cooperates well with continuity.

Theorem 1.6. *Suppose that $(f_n) \subset \mathcal{B}(S)$ is a sequence of continuous functions converging uniformly to $f : S \to \mathbf{R}$. Then f is continuous.*

To see that this theorem is special, note that it is false if we replace 'continuous' with 'differentiable' in the hypothesis and conclusion. Can you think of a counterexample?

Proof. Let $x \in S$ and $\epsilon > 0$ be given. Since $f_n \to f$ uniformly, there exists $N \in \mathbf{N}$ such that $\|f_n - f\| < \epsilon/3$ whenever $n \geq N$. Since f_N is continuous at x, we also have $\delta > 0$ such that $|f_N(y) - f_N(x)| < \epsilon/3$ whenever $y \in S$ and $|y - x| < \delta$. Thus

$$\begin{aligned} |g(y) - g(x)| &= |(g(y) - f_N(y)) + (f_N(y) - f_N(x)) + (f_N(x) - g(x))| \\ &\leq |g(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - g(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \end{aligned}$$

for all $y \in S$ such that $|y - x| < \delta$. Hence g is continuous at x. □

Finally, we show that uniform convergence cooperates well with integration.

Theorem 1.7. *Suppose that $(f_n) \subset \mathcal{B}(S)$ is a sequence of continuous functions converging uniformly to $f : S \to \mathbf{R}$. If $[a, b] \subset S$ is a closed and bounded interval, then*

$$\lim_{n \to \infty} \int_a^b f_n(t)\, dt = \int_a^b f(t)\, dt.$$

Proof. We have

$$\left| \int_a^b f_n(t)\, dt - \int_a^b f(t)\, dt \right| \leq \int_a^b |f_n(t) - f(t)|\, dt \leq \int_a^b \|f_n - f\|\, dt = (b - a) \|f_n - f\|.$$

Hence,

$$\lim_{n \to \infty} \left| \int_a^b f_n(t)\, dt - \int_a^b f(t)\, dt \right| \leq (b - a) \lim_{n \to \infty} \|f_n - f\| = 0,$$

since $f_n \to f$ uniformly on S. □

2. The existence and uniqueness theorem for first order ODEs

The fundamental fact about ordinary differential equations is that, under suitably nice circumstances and subject to appropriate initial conditions, one gets unique solutions. Here we will discuss this fact in the particular case of first order ODEs. The case of first order *systems* of ODEs is quite similar and essentially contains all other possible cases.

Let us set up the problem before stating any results. We begin with an open set $U \subset \mathbf{R}^2$ and a function $F : U \to \mathbf{R}$. Given any point $(t_0, y_0) \in U$ we seek solutions to the initial value problem

$$y'(t) = F(t, y(t)), \quad y(t_0) = y_0. \tag{1}$$

The domain of the function y is not so important here, so we allow ourselves to consider any differentiable function $y : I \to \mathbf{R}$ defined on an open interval I containing t_0. If such a y satisfies (1), then we refer to $y : I \to \mathbf{R}$ as a *solution* of (1). Note that different solutions can have different domains, but the domains of any two solutions must intersect in an open interval containing t_0.

Theorem 2.1 (Existence and Uniqueness Theorem)**.** *Suppose that $F = F(t, y)$ is continuous on U and furthermore continuously differentiable with respect to the second variable y. Then for any $(t_0, y_0) \in U$ there is a solution $y : I \to \mathbf{R}$ of the initial value problem* (1)*. This solution is unique in the following sense: if $\tilde{y} : \tilde{I} \to \mathbf{R}$ is another solution, then $\tilde{y}(t) = y(t)$ for all $t \in I \cap \tilde{I}$.*

Proof. Since U is open and $(t_0, y_0) \in U$, there exists a closed rectangle

$$R = [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b] \subset U.$$

A continuous function on such a rectangle will be bounded, so we have constants $A, B > 0$ such that

$$|F(t, y)| \leq A, \quad \left| \frac{\partial F}{\partial y}(t, y) \right| \leq B \quad \text{for all } (t, y) \in R.$$

In particular, by the mean value theorem, we have for all $(t, y_1), (t, y_2) \in R$

$$|F(t, y_1) - F(t, y_2)| = \left| \frac{\partial F}{\partial y}(t, c)(y_1 - y_2) \right| \leq B|y_1 - y_2|$$

where c is a number between y_1 and y_2.

Lemma 2.2. *Suppose $\epsilon \leq b$ is a positive number no larger than $\frac{b}{A}$. Let $I = (t_0 - \epsilon, t_0 + \epsilon)$ and $y : I \to [y_0 - b, y_0 + b]$ is a continuous function. Then the function $\tilde{y} : I \to \mathbf{R}$ given by*

$$\tilde{y}(t) = y_0 + \int_{t_0}^{t} F(s, y(s))\, ds$$

is well-defined and satisfies $\tilde{y}(t) \in [y_0 - b, y_0 + b]$ for all $t \in I$.

Proof. The hypothesis on y implies that $(t, y(t)) \in R$ for all $t \in I$. This, and continuity of y and F, imply that the right side of the equation defining $\tilde{y}$ makes sense for all $t \in I$. We have moreover for such t that

$$|\tilde{y}(t) - y_0| = \left| \int_{t_0}^{t} F(s, y(s))\, ds \right| \leq A|t - t_0| \leq A\epsilon \leq A\epsilon \leq b,$$

where the last inequality comes from the hypothesis on ϵ. $\square$

Continuing to let $I = (t_0 - \epsilon, t_0 + \epsilon)$, with $\epsilon > 0$ as in the lemma, we invoke the conclusion of the lemma to define a sequence of functions $y_n : I \to \mathbf{R}$, $n \in \mathbf{N}$ inductively as follows:

$$y_0(t) \equiv y_0, \quad y_{n+1}(t) = y_0 + \int_{t_0}^{t} F(s, y_n(s))\, ds.$$

We will show that (y_n) converges uniformly to a solution of (1). The key step in doing so is our next lemma. Note that $\|\cdot\|$ means $\|\cdot\|_I$ here.

Lemma 2.3. *Suppose that $\epsilon < 1/2B$. Then for all $n \geq 1$,*

$$\|y_{n+1} - y_n\| \leq \frac{1}{2} \|y_n - y_{n-1}\| .$$

In particular,

$$\|y_{n+1} - y_n\| \leq \frac{1}{2^n} \|y_1 - y_0\| .$$

Proof. For any $n \geq 1$ and $t \in I$, we have

$$\begin{aligned} |y_{n+1}(t) - y_n(t)| &= \left| \int_{t_0}^{t} (F(s, y_n(s)) - F(s, y_{n-1}(s)))\, ds \right| \leq B \left| \int_{t_0}^{t} |y_n(s) - y_{n-1}(s)|\, ds \right| \\ &\leq B|t - t_0|\, \|y_n - y_{n-1}\|\, | < B\epsilon \|y_n - y_{n-1}\| \leq \frac{1}{2} \|y_n - y_{n-1}\| \end{aligned}$$

Since $t \in I$ on the left side was arbitrary, the first conclusion of the lemma follows. The second conclusion is obtained by iterating the first:

$$\|y_{n+1} - y_n\| \leq \frac{1}{2} \|y_n - y_{n-1}\| \leq \frac{1}{4} \|y_{n-1} - y_{n-2}\| \leq \cdots \leq \frac{1}{2^n} \|y_1 - y_0\| .$$

□

From now on we assume that $\epsilon \leq \min\{a, b/A, 1/2B\}$ satisfies the hypotheses of both of the above lemmas. We demonstrate convergence of the sequence (y_n) be rewriting the nth term as a telescoping sum:

$$y_n(t) = y_0 + \sum_{j=1}^{n} y_j(t) - y_{j-1}(t).$$

Hence, the limit of the sequence (if it exists) may be written as a telescoping series

$$y(t) = \lim_{n \to \infty} y_n(t) = y_0 + \sum_{j=1}^{\infty} y_j(t) - y_{j-1}(t).$$

Note here that by the previous lemma, the jth term in this series satisfies

$$|y_j(t) - y_{j-1}(t)| \leq \|y_j - y_{j-1}\| \leq \frac{C}{2^j}$$

for $C = \|y_1 - y_0\|$. Since $\sum_{j=1}^{\infty} \frac{C}{2^j}$ converges, it therefore follows from the Weierstrass M-test that (y_n) converges uniformly to some function $y : I \to \mathbf{R}$. Uniform convergence implies that y is continuous, since all of the y_n are continuous. We have moreover that

Lemma 2.4. *For any $t \in I$,*

$$\lim_{n \to \infty} \int_{t_0}^{t} F(s, y_n(s))\, ds = \int_{t_0}^{t} F(s, y(s))\, ds$$

Proof. Given $t \in I$, we estimate as in the preceding lemma

$$\left| \int_{t_0}^{t} (F(s, y_n(s)) - F(s, y(s)))\, ds \right| \leq B|t - t_0| \, \|y_n - y\| \, .$$

Uniform convergence of y_n to y means exactly that $\lim_{n\to\infty} \|y_n - y\| = 0$. Hence the right side tends to zero as $n \to \infty$, and the lemma is proved. □

Now we claim that $y : I \to \mathbf{R}$ is a solution of (1). First of all,

$$y(t_0) = y_0 + \int_{t_0}^{t_0} F(s, y(s))\, ds = y_0,$$

so y satisfies the right initial condition. We further have from the previous lemma that

$$y(t) = \lim y_n(t) = y_0 + \lim_{n\to\infty} \int_{t_0}^{t} F(s, y_{n-1}(s))\, ds = y_0 + \int_{t_0}^{t} F(s, y(s)\, ds.$$

Applying the fundamental theorem of calculus to the integral on the right side, we therefore obtain

$$y'(t) = 0 + F(t, y(t)) = F(t, y(t))$$

for all $t \in I$. This proves our claim and concludes the existence portion of the proof.

Finally, we turn to uniqueness. Suppose that $\tilde{y} : \tilde{I} \to \mathbf{R}$ is another solution of (1). First we show that y and $\tilde{y}$ agree near t_0.

Lemma 2.5. *There exists an open interval $J \subset I \cap \tilde{I}$ containing t_0 such that $\tilde{y}(t) = y(t)$ for all $t \in J$.*

Proof. By continuity of $\tilde{y}$, there is an open interval $J \subset \tilde{I} \cap I$ containing t_0 such that $|\tilde{y}(t) - y_0| < b$ for all $t \in J$. That is, $\tilde{y}(t) \in [y_0 - b, y_0 + b]$. Also, by integrating both sides of $\tilde{y}' = F(t, \tilde{y})$, we obtain

$$\tilde{y}(t) = t_0 + \int_{t_0}^{t} F(s, \tilde{y}(s))\, ds$$

for all $t \in J$. Thus we may argue as in the proof of Lemma 2.3 that

$$|\tilde{y}(t) - y(t)| = \left| \int_{t_0}^{t} (F(s, \tilde{y}(s)) - F(s, y(s)))\, ds \right| \leq \frac{1}{2} \|\tilde{y} - y\| \, .$$

Since $t \in J$ is arbitrary, this implies that $\|\tilde{y} - y\| \leq \frac{1}{2} \|\tilde{y} - y\|$, which can only happen if $\|\tilde{y} - y\| = 0$—i.e. if $\tilde{y} \equiv y$ on J. □

To conclude the proof of uniqueness in Theorem 2.1, we let $J \subset I \cap \tilde{I}$ be the largest open interval containing t_0 on which y and $\tilde{y}$ agree. By the previous lemma, we know at least that J is not empty. We assume, in order to reach a contradiction, that $J \neq I \cap \tilde{I}$. Under this assumption, we have that one of the two endpoints t_1 of J lies in $I \cap \tilde{I}$. For the sake of definiteness, we assume that t_1 is the righthand (i.e. upper) endpoint of J.

By continuity of y and $\tilde{y}$, we have

$$y_1 := y(t_1) = \lim_{t\to t_1^-} y(t) = \lim_{t\to t_1^-} \tilde{y}(t) = \tilde{y}(t_1)$$

Hence $y : I \to \mathbf{R}$ and $\tilde{y} : \tilde{I} \to \mathbf{R}$ are also both solutions of (1) subject to the initial condition $y(t_1) = y_1$. By Lemma 2.5, it follows that $y \equiv \tilde{y}$ on an open interval $J_1 \subset I \cap \tilde{I}$ containing

t_1. Thus $J \cup J_1$ is an open interval strictly larger than J on which $\tilde{y} \equiv y$, contradicting the assumption that J is the largest interval of agreement about t_0. We conclude that $y \equiv \tilde{y}$ everywhere on $I \cap \tilde{I}$. □

In closing, let us observe that the uniqueness part of Theorem 2.1 ensures us that there is a solution $y : I_{max} \to \mathbf{R}$ for which the interval I_{max} is as large as possible. To see that this is so, let

$$I_{max} = \bigcup \{J \subset \mathbf{R} : J \text{ is an open interval about } t_0 \text{ on which (1) admits a solution}\}$$

be the union of all solution intervals. Then I_{max} is an open interval about t_0 (why?), and we can define our solution $y : I_{max} \to \mathbf{R}$ at any point $t \in I_{max}$ by setting $y(t) = \tilde{y}(t)$ where $\tilde{y} : J \to \mathbf{R}$ is a solution whose domain J contains t. Since solutions agree on the intersection of their domains, it will not matter which solution $\tilde{y}$ we use to define $y(t)$. We will have moreover that $y \equiv \tilde{y}$ on all of J, so that in particular $y' = F(t, y)$ holds at t. That is, y satisfies (1) on all of I_{max}. We have just shown

Theorem 2.6 (Existence of solutions with maximal domain)**.** *Under the hypotheses of Theorem 2.1, there exists a solution $y : I_{max} \to \mathbf{R}$ of (1) whose domain I_{max} contains the domains of all other solutions $\tilde{y} : I \to \mathbf{R}$ of (1).*

An important feature of the solution $y : I_{max} \to \mathbf{R}$ is that it persists until its graph 'exits' the domain of existence U for the righthand side $F(t, y)$ of (1). This can be stated more precisely in terms of 'compact sets'. A subset $K \subset \mathbf{R}^n$ is called *compact* if it is closed and bounded. So for instance, an interval $I \subset \mathbf{R}$ is compact if and only if $I = [a, b]$ where $a, b \in \mathbf{R}$.

Theorem 2.7. *Let $y : I_{max} \to \mathbf{R}$ be the solution of (1) with maximal domain I_{max}. If $K \subset U$ is any compact set, then there is a compact interval $I_K \subset I_{max}$ such that $(t, y(t)) \notin K$ for any $t \notin I_K$.*

The conclusion of this theorem is often summarized by saying that the graph of $y : I_{max} \to \mathbf{R}$ is a curve that is 'properly embedded' in U. The proof of Theorem 2.7 depends on a refinement of the existence and uniqueness theorem. The gist of the refinement is that solutions to (1) vary continuously with the initial condition. In particular, the intervals on which these solutions are defined may be taken to vary continuously with the initial condition.

Theorem 2.8 (Existence and uniqueness with stability)**.** *Under the hypotheses of Theorem 2.1, let $K \subset U$ be a compact subset. Then there exists $\epsilon > 0$ and a continuous function $\phi : K \times (-\epsilon, \epsilon) \to \mathbf{R}$ such that for any $(t_0, y_0) \in K$ the function $y : (t_0 - \epsilon, t_0 + \epsilon) \to \mathbf{R}$ given by $y(t) := \phi(t_0, y_0, t - t_0)$ is the unique solution of (1).*

This theorem follows from a slightly more careful version of the proof of existence used for Theorem 2.1. We omit the details here. For purposes of proving Theorem 2.7, the important part of the stability theorem is that it gives a positive lower bound ϵ on the 'lifespan' of any solution that begins in K.

Proof of Theorem 2.7. Suppose that the theorem is false for some compact set $K \subset U$. Write $I_{max} = (a, b)$ (note that either or both of a and b could be infinite). Let $\epsilon > 0$ be the constant associated to K in the Stability Theorem. Then taking the closed interval $J = [a + \epsilon, b - \epsilon] \subset I_{max}$, we may choose a point $t_1 \in I_{max} - J$ such that $(t_1, y_1) \in K$, where

$y_1 := y(t_1)$. Thus y is a solution of $y' = f(t,y)$ subject to the initial condition $y(t_1) = y_1$. The stability theorem guarantees us that there is another solution $\tilde{y} : (t_1 - \epsilon, t_1 + \epsilon) \to \mathbf{R}$ of the same initial value problem. By uniqueness, we therefore have $\tilde{y} \equiv y$ on $I_{max} \cap (t_0 - \epsilon, t_0 + \epsilon)$. By setting,

$$\hat{y}(t) = \begin{cases} y(t) & \text{if } t \in I_{max} \\ \tilde{y}(t) \text{ if } |t - t_1| < \epsilon, \end{cases}$$

we obtain that $\hat{y}$ is a solves (1) on $I_{max} \cup (t_1 - \epsilon, t_1 + \epsilon)$. Since this last interval is not contained in I_{max}, we have contradicted the fact that I_{max} is the maximal domain of existence for the solution of (1). So the theorem holds. □

3. Asymptotic Behavior of Solutions to Autonomous 1st Order Equations

In this section we consider solutions of the initial value problem

$$y' = f(y), \quad y(t_0) = y_0 \tag{2}$$

where $f : \mathbf{R} \to \mathbf{R}$ is a C^1 function. Differential equations like the one here, in which the right side does not depend explicitly on t, are called *autonomous*. Such ODEs are both common in applications and important in theory[1]

The ODE in (2) is separable and therefore in principle solvable by integration. In practice, however, the integration can be unmanageable and will only give an implicit and fairly unenlightening formula for the solution. Here we take a more qualitative approach to analyzing the problem, and in particular, understanding what happens to the solution as t tends toward the ends of the maximal domain of existence. This is, in pedestrian terms, somewhat akin to plugging information about today's weather into the equations of fluid mechanics to try and infer whether one ought to plan picnics in the year 10,000. In this light, it is somewhat remarkable that we will be able to say anything sensible at all.

If $f(y_0) = 0$ then we call y_0 an *equilibrium point* of the ODE. In this case, one checks easily that $y : \mathbf{R} \to \mathbf{R}$ given by $y(t) \equiv y_0$ is the solution of (2). In particular, the maximal domain of existence is all of $\mathbf{R}$, and we have $\lim_{t\to\pm\infty} y(t) = y_0$. If $f(y_0) \neq 0$, then things are certainly more complicated. For definiteness' sake, let us suppose from now on that $f(y_0) > 0$, and consider the solution $y : I \to \mathbf{R}$ of (2) with maximal domain of existence $I = (a, b)$.

Lemma 3.1. $y'(t) > 0$ *for all* $t \in I$.

Proof. If the assertion is false, then there exists $t_1 \in I$ such that $f(y(t_1)) = y'(t_1) \leq 0$. Since $f(y(t_0)) > 0$, the intermediate value theorem tells us there exists t_2 between t_0 and t_1 such that $f(y(t_2)) = 0$. But then $y(t_2) = z(t_2)$ where $z(t) \equiv y(t_2)$ is a constant solution. So by uniqueness of solutions to initial value problems, we see that $y(t) = z(t)$ for all $t \in I$. In particular $0 < f(y(t_0)) = f(y(t_2)) = 0$, which is a contradiction. So the assertion is true. □

Lemma 3.2. $\lim_{t\to b} y(t)$ *exists (and is possibly* ∞*).*

Proof. By the previous lemma y is an increasing function. Letting $L = \sup_{t\in I} y(t)$, we claim $\lim_{t\to b} y(t) = L$. We prove the claim only in the case $L = \infty$, leaving the case $L < \infty$ to you. Given $M \in \mathbf{R}$, we know there exists $T \in I$ such that $y(T) > M$. So if $T < t < b$, we

[1]In somewhat the same way we reduced solving higher order ODEs to solving first order systems, one can always reduce a non-autonomous ODE to a first order autonomous system.

see that $y(t) \geq y(T) > M$, because y is increasing. Since M is arbitrary, we conclude that $\lim_{t\to b} y(t) = \infty$. □

Lemma 3.3. *If $L = \lim_{t\to b} y(t) < \infty$, then $b = \infty$ and $f(L) = 0$.*

Proof. Let $L = \lim_{t\to b} y(t)$, and for any $M > t_0$ consider the compact set $K := [t_0, M] \times [y_0, L]$. Then by Theorem 2.7, there exists $T \in I$ such that $(t, y(t)) \notin K$ for all $t > T$. Since $(t_0, y_0) \in K$, it follows that $t_0 < T < b$. So if $T < t < b$, we have $y(t) \in [y_0, M]$. It follows therefore that $t > M$. Thus $b > t > M$ for any $M \in \mathbf{R}$, which means that $b = \infty$.

Now since f and y are continuous, we have $\lim_{t\to\infty} f(y(t)) = f(\lim_{t\to\infty} y(t)) = f(L)$. So given $\epsilon > 0$, we have $T \in \mathbf{R}$ such that $t > T$ implies $|f(y(t)) - f(L)| < \epsilon/2$. In fact, since $L \geq f(y(t))$, we have $0 \leq f(L) - f(y(t)) < \epsilon/2$.

So if we choose any $t > T$, the mean value theorem gives us $s \in (t, t+1)$ such that

$$f(y(s)) = \frac{f(y(t+1)) - f(y(t))}{(t+1) - t} < \frac{f(L) - (L - \epsilon/2)}{1} = \epsilon/2.$$

That is, there exists $s > T$ such that $0 < f(y(s)) < \epsilon$. Thus,

$$|f(L)| = |f(L) - f(s) + f(s)| \leq |f(L) - f(s)| + |f(s)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

□

Let us combine the assertions above into a single statement.

Theorem 3.4. *Suppose that $f(y_0) > 0$ and that $y : (a, b) \to \mathbf{R}$ is the solution of (2) with maximal domain of existence I. Then y is strictly increasing on all of (a, b), and there are two possibilities for the asymptotic behavior of $y(t)$ as t increases. Either*

- $\lim_{t\to b} y(t) = \infty$*; or*
- $b = \infty$ *and* $\lim_{t\to\infty} y(t) = L$ *where L is an equilibrium point of the ODE.*

We leave it to the reader to puzzle out the statement of this theorem in the case $f(y_0) < 0$ and to draw the appropriate conclusions about the asymptotic behavior of the global solution $y : I \to \mathbf{R}$ as t decreases toward the *left* endpoint of the domain I. In essence, Theorem 3.4 is telling us that solutions to first order autonomous ODEs will either drift off to infinity or settle down and become asymptotically constant. If the reader finds this unsurprising, then he or she should try to imagine what the analogous assertion should be for solutions of autonomous systems of 2 or 3 ODEs (Hint: don't even try when there are 3 or more equations involved.)